Global Journal of Mathematics and Statistics
ISSN: 3078-8927
DOI: 10.61424/gjms
Journal Homepage: www.bluemarkpublishers.com/index.php/GJMS

GJMS
BLUEMARK PUBLISHERS

| RESEARCH ARTICLE

Fuzzy Lagrange Polynomials for Solving Two-Dimensional Fuzzy Fractional Volterra Integro-Differential Equations

Rasha Hassein Ibraheem
Department of Mathematics, College of Basic Education, Al-Mustansiriyah University, Baghdad 14022, Iraq
Corresponding Author: Rasha Hassein Ibraheem, **E-mail**: rasha.hassen83.edbs@uomustansiriyah.edu.iq

| ABSTRACT
In this study, we present a numerical approach to solve first-order fuzzy fractional Volterra integro-differential equations in two dimensions space, using three different formulations of fuzzy Lagrange polynomials: the fuzzy original Lagrange polynomial (FOLP), the fuzzy barycentric Lagrange polynomial (FBLP), and the fuzzy modified Lagrange polynomial (FMLP). Comprehensive algorithm is constructed to improve the computational efficiency of the proposed method and its effectiveness was tested through numerical application the numerical results demonstrate that the three methods can preserve the basic properties of fuzzy solutions, with the FMLP method achieving superior performance in terms of accuracy in solving the two-dimensional fuzzy fractional integro-differential equation based on the lowest absolute error. To verify the effectiveness of the methods, the resulting numerical solutions were compared graphically with the exact solution.

| KEYWORDS

Two-dimensional fuzzy Fractional Volterra Integro-Differential Equations, fuzzy Lagrange polynomials

| ARTICLE INFORMATION

ACCEPTED: 21 April 2025 **PUBLISHED:** 06 July 2025 **DOI:** 10.61424/gjms.v2.i1.291

1. Introduction

Fuzzy fractional differential and integral equations are a prominent topic in fuzzy mathematics, due to their ability to represent complex models with uncertainties. This field has received extensive research attention [1–8], given its wide range of applications in various fields such as engineering, operations research, physics, and computer science. The importance of this theory lies in its ability to transform practical problems into mathematical formulas that can be analyzed within a precise fuzzy fractional framework by transforming them into uncertain systems of a fractional nature that can be mathematically treated. Historically, the roots of fractional calculus date back to 1823, when Abel first used fractional derivatives to solve an integral equation while studying the tautochrone problem. Integration and fractional derivatives have been used within the Riemann–Liouville form based on the Hokuhara differentiation framework to address and solve some mathematical problems in a specific field [9- 12]. Fractional calculus is used to model many phenomena related to scientific and engineering problems. Given the difficulty of solving most fuzzy fractional integro-differential equations analytically, finding approximate solutions using numerical methods is increasingly important. Fuzzy fractional integro-differential equations have emerged as a significant topic of study in recent scientific literature [13, 14]. Many researchers have focused on studying the properties of these equations and finding solutions using a variety of analytical methods. These methods include the fuzzy Laplace transform [15], the two-dimensional Legendre wavelet technique [16], the Adomian decomposition method [17], the variational iteration method [18], and the domain-appropriate Hilbert kernel technique [19]. The importance of these techniques lies in their ability to handle fuzzy data effectively and represent dynamical systems in natural environments characterized by ambiguity and uncertainty. These concepts are of particular importance within the theory of fuzzy analysis, as they have broad applications in diverse fields such as, fuzzy control models, quantum optics, gauge theory, atmospheric studies, and others [20–23]. In many real-world situations, the information associated with problems is always tainted with a degree of uncertainty, which arises from various factors, such as measurement errors, missing data, or unclear constraint conditions. Accordingly, it becomes necessary to deal with these uncertainties with appropriate methods to improve the results Recently, Lagrange methods have been employed to address fractional integro-differential equations [24], with Lagrange's fuzzy interpolation polynomial also being utilized for this purpose. In 2024, Ting Ding and Jin Huang [25] applied the Legendre spectral method for solving functional integro-differential equations. [26] Lagrangian spectral grouping has been proposed as a methodology for treating single weak-kernel fractional integral-differential equations. This study primarily addresses on the following form of equations:

$$D_x^{\mu}u(\mathrm{x},\mathrm{y};\alpha) = h(x,y) + \int_0^y \int_0^x k(x,y,z,t)G(u(z,t;\alpha))\,dz\,dt \qquad (1.1)$$

With the initial condition $\mathrm{u}_\alpha(0,\mathrm{y}) = [\mathrm{u}_{1\alpha}(0,\mathrm{y}), \mathrm{u}_{2\alpha}(0,\mathrm{y})]$,for all $\alpha, \mu \in [0,1]$,where $(x,y) \in I$, $I = [0,1] \times [0,1]$, D_x^{μ} is the fuzzy fractional derivative in caputo sense of order μ , $\mathrm{u}_\alpha(\mathrm{x},\mathrm{y})$ is a unknown analytic fuzzy function, and $h(x,y)$ is a continuous fuzzy valued function defined on the domain I ,$k(x,y,z,t)$ is a positive continuous real- valued kernel function of $I \times I$, $G(\mathrm{u}_\alpha(z,t))$ is a continuous fuzzy- valued function that is Lipschitz continuous.

The structure of the paper is as: Section 1 provides a general introduction, while Section 2 presents the basic concepts of calculus and fractional integration. Section 3 is devoted to the analysis of two-dimensional fuzzy fractional Volterra differential equations, while Section 4 presents a detailed derivation of the proposed methods. Section 5 includes a numerical application, and Section 6 concludes with the most important results and conclusions.

2. Fuzzy Calculus and Fuzzy Fractional Calculus

In this section, basic symbols, definitions, and main results related to fuzzy fractional calculus are presented. . A fuzzy number ũ can be characterized as a fuzzy subset of ℝ. For additional information, please consult references [27-33].

Definition1. [27] In the parametric form, a fuzzy number ũ is represented by a pair of functions$(\boldsymbol{u_1}(\alpha), \boldsymbol{u_2}(\alpha))$ defined for $\alpha \in (0,1]$, that satisfy the following properties

$u_1(\alpha)$ is a bounded non-decreasing function, left continuous for each $\alpha \in (0,1]$, and right continuous at $\alpha = 0$.

$u_2(\alpha)$ is a bounded non-increasing function, left continuous for each $\alpha \in (0,1]$, and right continuous at $\alpha = 0$.

$u_1(\alpha) \leq u_2(\alpha)$ on $[0,1]$.

Definition1. [27] The class of fuzzy subsets on real axis is denoted by parametric form$\mathrm{R_F} = \{\tilde{\mathrm{u}}: \mathrm{R} \to [0,1]\}$. If ũ is normal, fuzzy convex, upper semicontinuous and its closure, $cl\{\mathrm{x},\mathrm{y} \in \mathrm{R}, \tilde{\mathrm{u}}(x,y,\alpha) > 0\}$ is compact, then $\mathrm{R_F}$ is said to be the space of fuzzy numbers.

Definition3. [30] Let $\mathrm{u} = (\mathrm{a},\mathrm{b}) \to \mathrm{R_F}$ and $\mathrm{x} \in (\mathrm{a},\mathrm{b})$. We say that u is strongly generalized differentiable at x , if there exists an element $u'(\mathrm{x}) \in \mathrm{R_F}$ such that:

The H-differences $u(\mathrm{x}+\mathrm{h})$ - $u(\mathrm{x})$, $u(\mathrm{x})$ - $u(\mathrm{x}-\mathrm{h})$ exist, ∀ h > 0 sufficiently close to 0, and $\lim_{\mathrm{h}\to 0+} \frac{u(\mathrm{x}+\mathrm{h}) - u(\mathrm{x})}{\mathrm{h}} = u'(\mathrm{x}) = \lim_{\mathrm{h}\to 0+} \frac{u(\mathrm{x}) - u(\mathrm{x}-\mathrm{h})}{\mathrm{h}}$

The H-differences $u(\mathrm{x})$ - $u(\mathrm{x}+\mathrm{h})$, $u(\mathrm{x}-\mathrm{h})$- (x) exist, ∀ h > 0 sufficiently close to 0, and $\lim_{\mathrm{h}\to 0+} \frac{u(\mathrm{x}) - u(\mathrm{x}+\mathrm{h})}{\mathrm{h}} = u'(\mathrm{x}) = \lim_{\mathrm{h}\to 0+} \frac{u(\mathrm{x}-\mathrm{h}) - u(\mathrm{x}^*)}{\mathrm{h}}$.

Definition4. [33] Let $\boldsymbol{\mu} > \mathbf{0}$ The Riemann–Liouville partial fractional integral operator $\boldsymbol{I_x^{\mu}}$ of order $\boldsymbol{\mu}$,with respect to $\boldsymbol{x}$ is expressed as

$$I_x^{\mu} u_{1\alpha}(\mathrm{x},\mathrm{y};\alpha) = \frac{1}{\Gamma(\mu)} \int_0^x (x-z)^{\mu-1} u_{1\alpha}(z,y;\alpha)\, dz, \quad I_x^0 u_{1\alpha}(\mathrm{x},\mathrm{y};\alpha) = u_{1\alpha}(\mathrm{x},\mathrm{y};\alpha)$$

And

$$\boldsymbol{I_x^{\mu} u_{2\alpha}(\mathrm{x},\mathrm{y};\alpha) = \frac{1}{\Gamma(\mu)} \int_0^x (x-z)^{\mu-1} u_{2\alpha}(z,y;\alpha)\, dz}\ , \quad \boldsymbol{I_x^0 u_{2\alpha}(\mathrm{x},\mathrm{y};\alpha) = u_{2\alpha}(\mathrm{x},\mathrm{y};\alpha)}$$

Definition 5.[33] Let $\boldsymbol{\mu} > \mathbf{0}$ The partial fractional derivative of $\boldsymbol{D_x^{\mu}}$ of order $\boldsymbol{\mu}$ with respect to $\boldsymbol{x}$ in the Caputo sense is defined as:

$$D_x^{\mu} u_1(\mathrm{x},\mathrm{y};\alpha) = \frac{1}{\Gamma(n-\mu)} \int_0^x (x-z)^{n-\mu-1} \mathrm{u}_{1\alpha}'(z,y;\alpha)\, dz \quad , n-1 < \mu \le n\ ,\ n \in Z^+\ ,\quad x > a$$

$$D_x^{\mu} u_2(\mathrm{x},\mathrm{y};\alpha) = \frac{1}{\Gamma(n-\mu)} \int_0^x (x-z)^{n-\mu-1} \mathrm{u}_{2\alpha}'(z,y;\alpha)\, dz \quad , n-1 < \mu \le n\ ,\ n \in Z^+\ ,\quad x > a$$

In this paper, we only consider differentiable of order $0 \le \mu \le 1$ for fuzzy-valued function f, such that:

$$D_x^{\mu} u_1(\mathrm{x},\mathrm{y};\alpha) = \frac{1}{\Gamma(1-\mu)} \int_0^x \frac{u_{1\alpha}'(z,y;\alpha)}{(x-z)^{\mu}}\, dz$$

and

$$D_x^{\mu} u_2(\mathrm{x},\mathrm{y};\alpha) = \frac{1}{\Gamma(1-\mu)} \int_0^x \frac{u_{2\alpha}'(z,y;\alpha)}{(x-z)^{\mu}}\, dz$$

3. Analysis of Two Dimensional Fuzzy Fractional Volterra IDEs

In this section, an analysis of two-dimensional fuzzy fractional Volterra integro-differential equations (2DFFVIDEs) will be presented, focusing on their differentiability according to Caputo's definition of H as well as Riemannian integration. The initial value is assumed to be a fuzzy number, and the solution is a fuzzy function. In this context, integration and differentiation are treated as two independent fuzzy operations, requiring careful formulation of the problem to ensure their correct handling.

Now, for all $\alpha \in (0,1]$ and $a \le x \le b\,, c \le y \le d$ let the parametric from of the fuzzy function $h(x,y)$ *and* $u(\mathrm{x},\mathrm{y})$,as$[h(x,y;\alpha)] = [h_1(x,y;\alpha), h_2(x,y;\alpha)]$ and, $[u(x,y;\alpha)] = [u_1(x,y;\alpha)$

$, u_2(x,y;\alpha)]$ and $k(x,y,z,t)G(u(z,t;\alpha))$ is given by:

$k(x,y,z,t)G(u(z,t;\alpha)) = k(x,y,z,t)G(u_1(z,t;\alpha), u_2(z,t;\alpha)\ )$, in which

$$k(x,y,z,t)G(u_1(z,t;\alpha), u_2(z,t;\alpha)\ ) = \begin{cases} k(x,y,z,t)G\big(u_1(z,t;\alpha)\big)\,, k(x,y,z,t) \ge 0 \\ k(x,y,z,t)G\big(u_2(z,t;\alpha)\big), k(x,y,z,t) < 0 \end{cases}$$

The (n)-solution of two dimensional fuzzy fractional Volterra integro-differential equations (1.1) is a function y:$[a,b] \to \mathrm{R_F}$ that has Caputo $[(n) - \mu]$-differentiable and satisfies (1.1). To compute it, we perform the next algorithm.

Algorithm 1: To obtain the (n)-solution of the two dimensional FFVIDEs (1.1) are presented as:
If u(x,y) is Caputo [(1)- μ]-differentiable, we convert the eq's(1.1) to the following system:

$$\begin{aligned} D_x^{\mu} \mathrm{u}_1(\mathrm{x},\mathrm{y};\alpha) &= \mathrm{h}_1(x,y;\alpha) + \int_0^y \int_0^x k(x,y,z,t) G(u_1(z,t;\alpha))\, dz\, dt \\ D_x^{\mu} \mathrm{u}_2(\mathrm{x},\mathrm{y};\alpha) &= \mathrm{h}_2(x,y;\alpha) + \int_0^y \int_0^x k(x,y,z,t) G(u_2(z,t;\alpha))\, dz\, dt \end{aligned} \quad (3.1)$$

With initial conditions

$$\boldsymbol{u}(0,\mathrm{y},\alpha)=[\mathrm{u}_1(0,\mathrm{y},\alpha),\mathrm{u}_2(0,\mathrm{y},\alpha)] \qquad (3.2)$$

Then, do the following steps:

step1: solve the eq's (3.1) and (3.2) for $\mathrm{u}_{1\alpha}(\mathrm{x},\mathrm{y}),\mathrm{u}_{2\alpha}(\mathrm{x},\mathrm{y})$

step2: Ensure that $[\mathrm{u}_1(\mathrm{x},\mathrm{y};\alpha),\mathrm{u}_2(\mathrm{x},\mathrm{y};\alpha)]$ and $[D_x^{\mu}\mathrm{u}_1(\mathrm{x},\mathrm{y};\alpha),D_x^{\mu}\mathrm{u}_2(\mathrm{x},\mathrm{y};\alpha)]$ are valid level sets on [a, b] or on a partial interval in [a,b].

step4: construct a (1)- differentiable solution u(x,y) whose $\alpha-cut$ representation is$[\mathrm{u}_1(\mathrm{x},\mathrm{y};\alpha),\mathrm{u}_2(\mathrm{x},\mathrm{y};\alpha)]$.

4. Method solutions

To Solve Eq. (1.1) by using OLP, MLP and BLP, the derivations of these methods are detailed in this section as outlined below.

4.1 Two-Dimensional Fuzzy Original Lagrange Polynomial Method:

The Lagrange approach is the best and most reliable method for polynomial interpolation. Let us $\mathrm{u}(\mathrm{x},\mathrm{y}\,;\alpha)\in C_F\,(A=[a,b]\times[c,d]), C_F\,(A)$ represent the two-dimensional space using

$a=x_0<x_1<\cdots<x_n=b$ and $c=y_0<y_1<\cdots<y_m=d$. Following that $(x_i,y_j), 0\le i\le n,\ 0\le j\le m$ are $(n+1)(m+1)$ tensor product interpolation nodes on area $A=[a,b]\times[c,d]$. The generalized fundamental functions for the original fuzzy Lagrange interpolation$l_{n,i}(\mathrm{x};\alpha)$ and $l_{mj}(\mathrm{y};\alpha)$ are derived from the studies in [35]. The original two-dimensional fuzzy Lagrange interpolation polynomial takes the following form:

$$\mathrm{p}_{n,m}\,(\mathrm{x},\mathrm{y}\,;\alpha)=\sum_{i=0}^{n}\sum_{j=0}^{m}\mathrm{u}(x_i,y_j;\alpha)\,l_{n,i}\,(\mathrm{x}\,;\alpha)l_{m,j}(\mathrm{y}\,;\alpha) \qquad (4.1)$$

$$such\ that\quad l_{n,i}(\mathrm{x};\alpha)=\prod_{\substack{k=0\\ i\neq k}}^{n}\frac{(x-x_k)}{(x_i-x_k)}\ \ and\ \ l_{m,j}(\mathrm{y};\alpha)=\prod_{\substack{l=0\\ j\neq l}}^{m}\frac{(y-y_l)}{(y_j-y_l)}$$

Now, for $x=x_i\ \ ,y=y_j$ then:

$$\mathrm{p}_{n,m}(x_i,y_j;\alpha)=\mathrm{u}(x_i,y_j;\alpha)=u_{i,j,\alpha}\qquad,\forall\, i=0,2,\ldots.,n,\quad \forall\, j=0,2,\ldots.,m \qquad (4.2)$$

Which that is mean:

$$l_{n,i}(x_i;\alpha)=\begin{cases}1 & if\,x=x_i\\ 0 & if\ \ x=x_k\ \ ,(i\neq k)\end{cases}\qquad ,l_{m,j}(y_j;\alpha)=\begin{cases}1 & if\,y=y_j\\ 0 & if\ \ y=y_l\ \ (j\neq l)\end{cases} \qquad (4.3)$$

And the dervitive Eq.(4.1) of order μ with respect to x we obtain:

$$D_x^{\mu}\mathrm{p}_{n,m}\,(\mathrm{x},\mathrm{y}\,;\alpha)=\sum_{i=0}^{n}\sum_{j=0}^{m}u_{i,j,\alpha}\,l_{n,i}^{\mu}{}'\,(\mathrm{x}\,;\alpha)l_{m,j}(\mathrm{y}\,;\alpha) \qquad (4.4)$$

In order to solve 2-dimensional FFVIDE using fuzzy Original Lagrange polynomial, we substituting Eq. (4.1)and Eq. (4.4)in Eq. (1.1), to get:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}u_{i,j,\alpha}\,l_{n,i}^{\mu}{}'\,(\mathrm{x}\,;\alpha)\,l_{m,j}(\mathrm{y}\,;\alpha)=\qquad h(x,y;\alpha)+\int_0^y\int_0^x k(x,y,z,t)G\left(\sum_{i=0}^{n}\sum_{j=0}^{m}u_{i,j,\alpha}\,\mathrm{l}_{n,i}\,(\mathrm{z}\,;\alpha)\,\mathrm{l}_{m,j}(\mathrm{t}\,;\alpha)\right)dz\,dt \qquad (4.5)$$

According to the definition of Caputo's derivative, it is expressed as:

$$l_{n,i}^{\mu}{}'(\mathrm{x};\alpha)=\frac{1}{\Gamma(1-\mu)}\int_0^x(x-z)^{-\mu}\,l'_{n,i}(\mathrm{z}\,;\alpha)dz=\frac{1}{\Gamma(1-\mu)}\int_0^x(x-z)^{-\mu}\left(\prod_{\substack{k=0\\ k\neq i}}^{n}\frac{(z-x_k)}{(x_i-x_k)}\right)'dz \qquad (4.6)$$

Therefore, after collecting the coefficients of $u_{k,l},\ k=0,1,\ldots.,n\ ,\ \ l=0,1,\ldots,m\ ,$ we obtain:

$$h(x,y;\alpha) = u_{0,0,\alpha}\left(l^{\mu}_{n,0}{}'(\mathrm{x};\alpha)l_{m,0}(\mathrm{y};\alpha) - \int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{l}_{n,0}(\mathrm{z};\alpha)\,\mathrm{l}_{m,0}(\mathrm{t};\alpha)\right)dz\,dt\right)$$
$$+u_{1,1,\alpha}\left(l^{\mu}_{n,1}{}'(\mathrm{x};\alpha)l_{m,1}(\mathrm{y};\alpha) - \int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{l}_{n,1}(\mathrm{z};\alpha)\,\mathrm{l}_{m,1}(\mathrm{t};\alpha)\right)dz\,dt\right)$$
$$+u_{2,2,\alpha}\left(l^{\mu}_{n,2}{}'(\mathrm{x};\alpha)l_{m,2}(\mathrm{y};\alpha) - \int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{l}_{n,2}(\mathrm{z};\alpha)\,\mathrm{l}_{m,2}(\mathrm{t};\alpha)\right)dz\,dt\right)$$
$$+u_{n,m,\alpha}\left(l^{\mu}_{n,n}{}'(\mathrm{x};\alpha)l_{m,m}(\mathrm{y};\alpha) - \int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{l}_{n,n}(\mathrm{z};\alpha)\,\mathrm{l}_{m,m}(\mathrm{t};\alpha)\right)dz\,dt\right)$$

Putting $x = x_i\,, y = y_j$, for $i = 0,2,\ldots.,n$, $j = 0,1,\ldots,m$ a system of n equations is formed as:

$$D\vec{U} = \vec{C} \qquad (4.7)$$

When

$$D = d_{i,j}\ ,\quad \vec{C} = C_i \ and\ \vec{U} = [u_{1,1,\alpha}, u_{2,2,\alpha}, .., u_{n,m,\alpha}]^T$$

with

$$C_i = h(x_i, y_j;\alpha) - u_{0,0,\alpha}(l^{\mu}_{n,1}{}'(x_i;\alpha)l_{m,1}(y_j;\alpha) - \int_0^{y_i}\int_0^{x_i} k(x_i,y_j,z,t)G\left(\mathrm{l}_{n,1}(\mathrm{z};\alpha)\,\mathrm{l}_{m,1}(\mathrm{t};\alpha)\right)dz\,dt \qquad (4.8)$$

And

$$d_{i,j} = l^{\mu}_{n,i}{}'(x_i;\alpha)l_{m,j}(y_j;\alpha) - \int_0^{y_j}\int_0^{x_i} k(x_i,y_j,z,t)G\left(\mathrm{l}_{n,i}(\mathrm{z};\alpha)\,\mathrm{l}_{m,j}(\mathrm{t};\alpha)\right)dz\,dt \qquad (4.9)$$

for $i = 0,2,\ldots.,n$, $j = 0,1,\ldots,m$

4.2 Two-Dimensional Fuzzy Barycentric Lagrange Polynomial Method:

The fuzzy valued function $\mathrm{p}_{n,m}(\mathrm{x},\mathrm{y};\alpha)$ can be represented using the two-dimensional fuzzy Barycentric Lagrange polynomial as:

$$\mathrm{p}_{n,m}(\mathrm{x},\mathrm{y};\alpha) = \sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j,\alpha}\, B_{n,i}(\mathrm{x};\alpha)B_{m,j}(\mathrm{y};\alpha) \qquad (4.10)$$

$$such\ that\quad B_{n,i}(\mathrm{x};\alpha) = \frac{\frac{w_i}{x-w_i}}{\sum_{k=0}^{n}\frac{w_k}{x-w_k}}\qquad ,B_{m,j}(\mathrm{y};\alpha) = \frac{\frac{v_j}{y-v_j}}{\sum_{l=0}^{m}\frac{v_l}{y-v_l}} \qquad (4.11)$$

Where

$$w_i = \frac{1}{\prod_{i\neq k}^{n}(x_i - x_k)},\qquad v_j = \frac{1}{\prod_{j\neq l}^{m}(y_j - y_l)} \qquad (4.12)$$

And the dervitive Eq.(4.10) of order μ with respect to x we obtain:

$$D_x^{\mu}\mathrm{p}_{n,m}(\mathrm{x},\mathrm{y};\alpha) = \sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j,\alpha}\, B^{\mu}_{n,i}{}'(\mathrm{x};\alpha)\,\mathrm{B}_{m,j}(\mathrm{y};\alpha) \qquad (4.13)$$

To solve two-dimensional FFVIDE using fuzzy Barycentric Lagrange polynomial, we substitute Eq. (4.10)and Eq. (4.13)in Eq. (1.1), to get:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j,\alpha}\, B^{\mu}_{n,i}{}'(\mathrm{x};\alpha)\, B_{m,j}(\mathrm{y};\alpha) = h(x,y;\alpha) + \int_0^y\int_0^x k(x,y,z,t)G\left(\sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j,\alpha}\,\mathrm{B}_{n,i}(\mathrm{z};\alpha)\,\mathrm{B}_{m,j}(\mathrm{t};\alpha)\right)dz\,dt$$
(4.14)

According to the definition of Caputo's derivative, it is expressed as:

$$B_{n,i}^{\mu}{}'(\mathrm{x};\alpha)=\frac{1}{\Gamma(1-\mu)}\int_0^x(x-z)^{-\mu}\,\mathrm{B}'_{n,i}(\mathrm{z}\,;\alpha)dz=\frac{1}{\Gamma(1-\mu)}\int_0^x(x-z)^{-\mu}\,\Big(\frac{\frac{w_i}{z-w_i}}{\sum_{k=0}^{n}\frac{w_k}{z-w_k}}\Big)'dz \qquad (4.15)$$

Therefore, after collecting the coefficients of $u_{k,l}$, $k=0,2,\dots,n$, $l=0,1,\dots,m$, we obtain:

$$h(x,y;\alpha)=u_{0,0,\alpha}\left(B_{n,o}^{\mu}{}'(\mathrm{x}\,;\alpha)B_{m,0}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{B}_{n,0}(\mathrm{z}\,;\alpha)\ \mathrm{B}_{m,0}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

$$+u_{1,1,\alpha}\left(B_{n,1}^{\mu}{}'(\mathrm{x}\,;\alpha)B_{m,1}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{B}_{n,1}(\mathrm{z}\,;\alpha)\ \mathrm{B}_{m,1}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

$$u_{2,2,\alpha}\left(B_{n,2}^{\mu}{}'(\mathrm{x}\,;\alpha)B_{m,2}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{B}_{n,2}(\mathrm{z}\,;\alpha)\ \mathrm{B}_{m,2}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

$$u_{n,m,\alpha}\left(B_{n,n}^{\mu}{}'(\mathrm{x}\,;\alpha)B_{m,m}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{B}_{n,n}(\mathrm{z}\,;\alpha)\ \mathrm{B}_{m,m}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

Putting $x=x_i\,, y=y_j$, for $i=0,2,\dots,n$, $j=0,1,\dots,m$, a system of n equations is formed as:

$$D\vec{U}=\vec{C} \qquad (4.16)$$

When

$$D=d_{i,j}\ ,\quad \vec{C}=C_i \ and\ \ \vec{U}=[u_{1,1,\alpha},u_{2,2,\alpha},..,u_{n,m,\alpha}]^T$$

with

$$C_i=h(x_i,y_j;\alpha)-u_{0,0,\alpha}(B_{n,1}^{\mu}{}'(x_i\,;\alpha)B_{m,1}(y_j\,;\alpha)-$$
$$\int_0^{y_i}\int_0^{x_i}k(x_i,y_j,z,t)G\left(\ \mathrm{B}_{n,1}(\mathrm{z}\,;\alpha)\ \mathrm{B}_{m,1}(\mathrm{t}\,;\alpha)\right)dz\,dt \qquad (4.17)$$

And

$$d_{i,j}=B_{n,i}^{\mu}{}'(x_i\,;\alpha)B_{m,j}(y_j\,;\alpha)-\int_0^{y_j}\int_0^{x_i}k(x_i,y_j,z,t)G\left(\mathrm{B}_{n,i}(\mathrm{z}\,;\alpha)\ \mathrm{B}_{m,j}(\mathrm{t}\,;\alpha)\right)dz\,dt \qquad (4.18)$$

for $i=0,2,\dots,n$, $j=0,1,\dots,m$

4.3 Two-Dimensional Fuzzy Modified Lagrange Polynomial Method:

The form of the two-dimensional fuzzy modified Lagrange interpolation polynomial is

$$\mathrm{p}_{n,m}\,(\mathrm{x},\mathrm{y}\,;\alpha)=\sum_{i=0}^{n}\sum_{j=0}^{m}\mathrm{u}(x_i,y_j;\alpha)\,w_i(x;\alpha)w_j(y;\alpha)\mathrm{M}_{n,i}\,(\mathrm{x}\,;\alpha)\ \mathrm{M}_{m,j}(\mathrm{y}\,;\alpha) \qquad (4.19)$$

Such that:

$$\mathrm{M}_{n,i}(\mathrm{x};\alpha)=l(x;\alpha)\frac{1}{\sum_{i=0}^{n}x-x_i}\ ,where\ \ l(x;\alpha)=\prod_{\substack{k=0\\ i\neq k}}^{n}(x-x_k)\ \ and\ \ \ w_i(x;\alpha)=\frac{1}{\prod_{i\neq k}^{n}(x_i-x_k)} \qquad (4.20)$$

$$\mathrm{M}_{m,j}(\mathrm{y};\alpha)=l(y;\alpha)\frac{1}{\sum_{j=0}^{m}y-y_j}\ ,where\ \ \ l(y;\alpha)=\prod_{\substack{l=0\\ j\neq l}}^{m}(y-y_l)\quad and\ \ w_j(y;\alpha)=\frac{1}{\prod_{j\neq l}^{m}(y_j-y_l)} \qquad (4.21)$$

And the dervitive Eq.(4.19) of order μ with respect to x we obtain:

$$D_x^{\mu}\mathrm{p}_{n,m}(\mathrm{x},\mathrm{y}\,;\alpha)=\sum_{i=0}^{n}\sum_{j=0}^{m}\mathrm{u}(x_i,y_j;\alpha)\,w_i(x;\alpha)w_j(y;\alpha)\,m_{n,i}^{\mu}{}'\,(\mathrm{x}\,;\alpha)\ \mathrm{M}_{m,j}(\mathrm{y}\,;\alpha) \qquad (4.22)$$

To solve the two-dimensional FFVIDE using fuzzy modified Lagrange polynomial, we substitute Eq. (4.19) and Eq. (4.22) in Eq. (1.1), to get:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}u_{i,j,\alpha}\,w_i(x;\alpha)w_j(y;\alpha)\ \mathrm{M}_{n,i}^{\mu}{}'\,(\mathrm{x}\,;\alpha)\ \mathrm{M}_{m,j}(\mathrm{y}\,;\alpha)=h(x,y;\alpha)+$$
$$\int_0^y\int_0^x k(x,y,z,t)G\big(\sum_{i=0}^{n}\sum_{j=0}^{m}u_{i,j,\alpha}\,w_i(x;\alpha)w_j(y;\alpha)\mathrm{M}_{n,i}\,(\mathrm{x}\,;\alpha)\ \mathrm{M}_{m,j}(\mathrm{y}\,;\alpha)\big)\,dz\,dt \qquad (4.23)$$

According to the definition of Caputo's derivative, it is expressed as:

$$\mathrm{M}_{n,i}^{\mu}{}'(\mathrm{x};\alpha)=\frac{1}{\Gamma(1-\mu)}\int_0^x (x-z)^{-\mu}\,\mathrm{M}'_{n,i}(\mathrm{z}\,;\alpha)dz=\frac{1}{\Gamma(1-\mu)}\int_0^x (x-z)^{-\mu}\left(l(z;\alpha)\frac{1}{\sum_{i=0}^{n} z-x_i}\right)'dz \qquad (4.24)$$

Therefore, after collecting the coefficients of $u_{k,l},\ k=0,2,\ldots.,n\ ,\ \ l=0,1,\ldots,m\ ,$ we obtain:

$$h(x,y;\alpha)=\ w_0(x;\alpha)w_0(y;\alpha)u_{0,0,\alpha}\left(\mathrm{M}_{n,0}^{\mu}{}'(\mathrm{x}\,;\alpha)\mathrm{M}_{m,0}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{M}_{n,0}(\mathrm{z}\,;\alpha)\,\mathrm{M}_{m,0}(\mathrm{t}\,;\alpha)\right)dz\,dt\ \right)$$

$$+w_1(x;\alpha)w_1(y;\alpha)u_{1,1,\alpha}\left(\mathrm{M}_{n,1}^{\mu}{}'(\mathrm{x}\,;\alpha)\mathrm{M}_{m,1}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{M}_{n,1}(\mathrm{z}\,;\alpha)\,\mathrm{M}_{m,1}(\mathrm{t}\,;\alpha)\right)dz\,dt\ \right)$$

$$+\,w_2(x;\alpha)w_2(y;\alpha)u_{2,2,\alpha}\left(M_{n,2}^{\mu}{}'(\mathrm{x}\,;\alpha)\mathrm{M}_{m,2}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{M}_{n,2}(\mathrm{z}\,;\alpha)\,\mathrm{M}_{m,2}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

$$+\,w_n(x;\alpha)w_m(y;\alpha)u_{n,m,\alpha}\left(M_{n,n}^{\mu}{}'(\mathrm{x}\,;\alpha)\mathrm{M}_{m,m}(\mathrm{y}\,;\alpha)-\int_0^y\int_0^x k(x,y,z,t)G\left(\mathrm{M}_{n,n}(\mathrm{z}\,;\alpha)\mathrm{M}_{m,m}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

Putting$x=x_i\,,y=y_j,$ for $=0,2,\ldots.,n\ ,\ \ j=0,1,\ldots,m$, a system of n equations is formed as:

$$D\vec{U}=\vec{C} \qquad (4.25)$$

When

$$D=d_{i,j}\ \ ,\quad \vec{C}=C_i\ and\ \ \vec{U}=[u_{1,1,\alpha},u_{2,2,\alpha},..,u_{n,m,\alpha}]^T$$

with

$$C_i=\qquad h(x_i,y_j;\alpha)-w_0(x_i;\alpha)w_0(y_j;\alpha)u_{0,0,\alpha}\left(M_{n,1}^{\mu}{}'(x_i\,;\alpha)\mathrm{M}_{m,1}(y_j\,;\alpha)-\int_0^{y_j}\int_0^{x_i} k(x_i,y_j,z,t)G\left(\mathrm{M}_{n,1}(\mathrm{z}\,;\alpha)\,\mathrm{M}_{m,1}(\mathrm{t}\,;\alpha)\right)dz\,dt\right) \qquad (4.26)$$

And

$$d_{i,j}=\ w_i(x_i;\alpha)w_j(y_j;\alpha)\left(M_{n,i}^{\mu}{}'(x_i\,;\alpha)\mathrm{M}_{m,j}(y_j\,;\alpha)-\int_0^{y_j}\int_0^{x_i} k(x_i,y_j,z,t)G\left(\mathrm{M}_{n,i}(\mathrm{z}\,;\alpha)\,\mathrm{M}_{m,j}(\mathrm{t}\,;\alpha)\right)dz\,dt\right)$$

for $i=0,2,\ldots.,n\ ,\ \ j=0,1,\ldots,m$ (4.27)

General Algorithm for the Proposed Methods

To compute the numerical solutions of the two-dimensional fuzzy fractional Volterra integro-differential equation using(FOLP),(FBLP), and(FMLP) methods, the following structured steps is employed:

Step1: Specify the step sizes as: $h=\frac{b-a}{n}\ \ ,z=\frac{d-c}{m}\quad ,n\in N\ ,m\in M,u(0,\mathrm{y};\alpha)=[\mathrm{u}_1(0,\mathrm{y};\alpha),\mathrm{u}_2(0,\mathrm{y};\alpha)]$

Step 2: Construct a uniform grid over the domain $[a,b]\times[c,d]$ as: $\mathrm{x}_i=a+ih$ with $\mathrm{x}_0=a,\ \mathrm{x}_n=b\ ,i=0,\ldots.,n$ $\&\,put\ \ \mathrm{y}_j=c+jz$ with $\mathrm{y}_0=c,\mathrm{y}_m=d\ ,j=0,\ldots.,m$

Step 3: According to steps (1) and (2).wefind the value of the linear system $D\vec{U}=\vec{C}$ and consider the following cases:

Case (1): Use Equations (4.8) and (4.9) associated with original Lagrange polynomial.

Case (2): Use Equations (4.17) and (4.18) associated with the barycentric Lagrange polynomial.

Case (2): Use Equations (4.26) and (4.27) associated with the modified Lagrange polynomial.

(Note that in all cases, the exact values of the fractional derivative and fractional integral are used according to Caputo's definition, which are calculated using MATLAB).

Step 4: Solve the system ($D\vec{U}=\vec{C}$) based on what was obtained in step 3 using Gauss's elimination method.

5. Numerical Example

In this section, we present a numerical example to illustrate the application of the methods discussed previously in solving a first-order two-dimensional fuzzy fractional Volterra integro-differentials Equation. The exact solution is known and serves as a reference to validate the accuracy of the numerical results obtained using our approaches. MATLAB version 7.6 was utilized to solve this example. The numerical accuracy is determined by the following errors:

$e_1(x_i, y_j; \alpha) = |u_1(x_i, y_j; \alpha) - p_{n,m,1}(x_i, y_j; \alpha)|$, $e_2(x_i, y_j; \alpha) = |u_2(x_i, y_j; \alpha) - p_{n,m,2}(x_i, y_j; \alpha)|$,

$$E_1(x_n, y_m; \alpha) = \max_{0\le x,y\le 1} e_1(x_i, y_j; \alpha) \;, E_2(x_n, y_m; \alpha) = \max_{0\le x,y\le 1} e_2(x_i, y_j; \alpha),$$

Example: Consider two-dimensional fuzzy fractional Volterra- integro-differential equations.

$$D_x^{0.75} u(\mathrm{x}, \mathrm{y}; \alpha) = h(x, y; \alpha) + \int_0^y \int_0^x (y+t) G(u(z, t; \alpha))\, dz\, dt$$

$$u(0, \mathrm{y}; \alpha) = [\mathrm{u}_1(0, \mathrm{y}; \alpha), \mathrm{u}_2(0, \mathrm{y}; \alpha)] = [\alpha^2 + \alpha, 4 - \alpha^4].$$

where

$$h_1(x, y; \alpha) = (\alpha^2 + \alpha)\frac{6.4}{\Gamma(0.25)} y x^{\frac{5}{4}} - (\alpha^2 + \alpha)\frac{5}{18} x^4 y^4$$

$$h_2(x, y; \alpha) = (4 - \alpha^4)\frac{6.4}{\Gamma(0.25)} y x^{\frac{5}{4}} - (4 - \alpha^4)\frac{5}{18} x^4 y^4$$

And the exact solution $u(\mathrm{x}, \mathrm{y}; \alpha) = [u_1(\mathrm{x}, \mathrm{y}; \alpha), u_2(\mathrm{x}, \mathrm{y}; \alpha)] = [\alpha^2 + \alpha, 4 - \alpha^4] x^2 y$

Tables 1 and 3 show the numerical comparison of the absolute errors obtained using the OLP, BLP, and MLP methods for N=M=2 and $\alpha = 0.2, 0.8$ respectively. Table 2 and 4 contains the maximum errors values for the same methods for N=M= 4, 5, and 8 and $\alpha = 0.2, 0.8$ respectively. The numerical solutions for the two functions are plotted and compared to the exact solutions in Figure 1.

Table1. Comparison of Absolute Error in Example (1) using OLP, BLP, and MLP methods with N=M=2. $\alpha = 0.2$

(x, y)	OLP $e_1(x_i, y_j; \alpha)$, $e_2(x_i, y_j; \alpha)$	BLP $e_1(x_i, y_j; \alpha)$, $e_2(x_i, y_j; \alpha)$	MLP $e_1(x_i, y_j; \alpha)$, $e_2(x_i, y_j; \alpha)$
(0.1,0.1)	0.000228 , 0.0037980	0.0023760 , 0.0039580	0.0002395 , 0.0039900
(0.4,0.2)	0.003648 , 0.0607757	0.0038016 , 0.0633347	0.0038323 , 0.0638465
(0.7,0.3)	0.014364 , 0.2393042	0.0149688, 0.24938020	0.0150897, 0.2513958
(0.2,0.4)	0.007296 , 0.1215514	0.0076032 , 0.1266693	0.0076646 , 0.1276922
(0.5,0.5)	0.028500 , 0.4748100	0.0297000 , 0.4948020	0.0299400 , 0.4988004
(0.8,0.6)	0.065664 , 1.0940534	0.0684288 , 1.1401189	0.0689822 , 1.1493328
(0.3,0.7)	0.033516 , 0.5580437	0.0349272 , 0.5815402	0.0352094 , 0.5862395
(0.6,0.8)	0.087552 , 1.4592547	0.0912384 , 1.5206970	0.0919757 , 1.5329855
(0.9,0.9)	0.166212 , 2.7691694	0.1732104 , 2.8857660	0.1745901 , 2.9080853
(1.0,1.0)	0.228000 , 3.7984800	0.2376000 , 3.9584160	0.2395200 , 3.9904032

Table 2. Maximum Error of Example (1) using OLP, BLP, and MLP methods with N=M=4, 5, 8 , $\alpha = 0.2$.

N,M	OLP $e_1(x_i, y_j; \alpha)$ $e_2(x_i, y_j; \alpha)$	BLP $e_1(x_i, y_j; \alpha)$ $e_2(x_i, y_j; \alpha)$	MLP $e_1(x_i, y_j; \alpha) e_2(x_i, y_j; \alpha)$
N=M=4	3.2477e-16,1.2932e-10	3.2477e-16,1.2932e-10	1.2932e-10, 1.2932e-10
N=M=5	5.1000e-16, 5.1000e-16	5.1000e-16, 5.1000e-16	5.1000e-16, 5.1000e-16
N=M=8	4.1440e-15 ,4.1440e-15	2.6173e-15, 2.6173e-15	2.6173e-15, 2.6173e-15

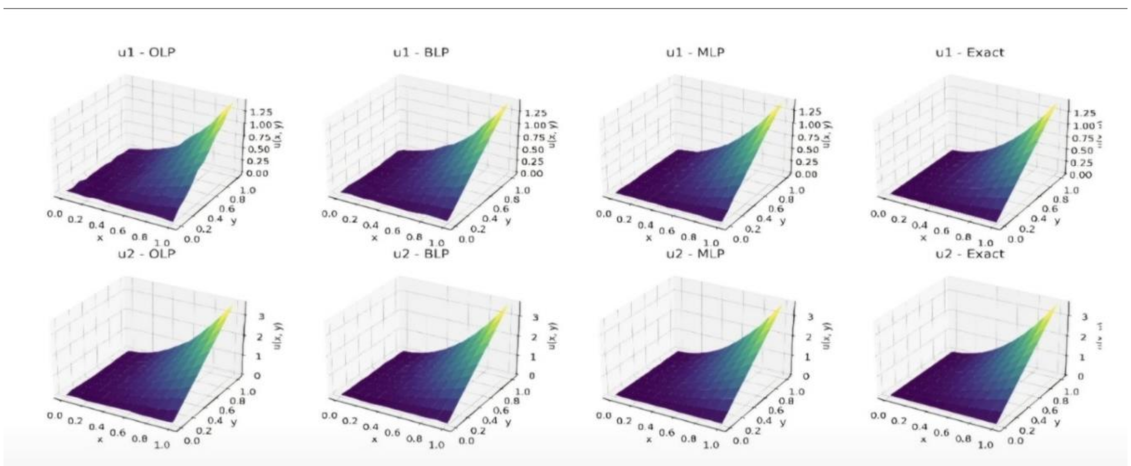
Table 3. Comparison of Absolute Error in Example (1) using OLP, BLP, and MLP methods with N=M=2. $\alpha = 0.8$

(x, y)	OLP $e_1(x_i, y_j; \alpha)$, $e_2(x_i, y_j; \alpha)$	BLP $e_1(x_i, y_j; \alpha)$, $e_2(x_i, y_j; \alpha)$	MLP $e_1(x_i, y_j; \alpha)$, $e_2(x_i, y_j; \alpha)$
(0.1,0.1)	0.001212 , 0.000208	0.000936 , 0.0003680	0.001201 , 0.000400
(0.4,0.2)	0.042432 , 0.054117	0.042278 , 0.051558	0.042248 , 0.051046
(0.7,0.3)	0.197316 , 0.288485	0.196711, 0.278409	0.196590, 0.276393
(0.2,0.4)	0.015744 , 0.064105	0.015437 , 0.069223	0.015375 , 0.070246
(0.5,0.5)	0.151500 , 0.026010	0.150300 , 0.046000	0.150060 , 0.050000
(0.8,0.6)	0.487296 , 0.284660	0.484531 , 0.238595	0.483978, 0.229381
(0.3,0.7)	0.057204 , 0.331849	0.055793 , 0.355345	0.055511 , 0.360044
(0.6,0.8)	0.327168 , 0.425220	0.323482 , 0.486662	0.322744 , 0.498950
(0.9,0.9)	0.883548 , 0.151768	0.876550 , 0.268364	0.875170 , 0.290684
(1.0,1.0)	1.212000 , 0.208080	1.202400 , 0.368016	1.200480 , 0.400003

Table 4. Maximum Error of Example (1) using OLP, BLP, and MLP methods with N=M=4, 5 , 8, $\alpha = 0.8$.

N,M	OLP $e_1(x_i, y_j; \alpha)$ $e_2(x_i, y_j; \alpha)$	BLP $e_1(x_i, y_j; \alpha)$ $e_2(x_i, y_j; \alpha)$	MLP $e_1(x_i, y_j; \alpha) e_2(x_i, y_j; \alpha)$
N=M=4	0.0239 ,0.0378	0.0085, 0.0220	0.0266 , 0.0112
N=M=5	0.0201 ,0.0296	0.0093 , 0.0294	0.0250, 0.0102
N=M=8	0.0298 ,0.0352	0.0113,0.0305	0.0419,0.0113

Figure1. Numerical and Exact Solution for $u_1(x, y; \alpha), u_2(x, y; \alpha)$ using OLP, BLP, and MLP Methods with $N = M = 10$ and $\alpha = 0.8$.



6. Conclusions

In this work, three types of two-dimensional fuzzy Lagrange polynomials are used to solve two-dimensional first–order fractional volterra integro differential equations: fuzzy original Lagrange polynomials, fuzzy barycentric Lagrange polynomial , and fuzzy modified Lagrange polynomial . Based on numerical results obtained from illustrative example, the following conclusions can be drawn:

- The results on absolute errors obtained using MATLAB confirm the effectiveness and reliability of the proposed approach.

- The modified Lagrange polynomial achieves a higher level of accuracy compared to other types of polynomials used.

- As the number of nodes (N, M) increases, the error decreases for all methods used.

- The method can be generalized to apply to nonlinear cases of fuzzy fractional Volterra equations.

- Its application can also be extended to solve n-order two-dimensional fuzzy FFVIDE equations.

Acknowledgments: The authors would like to thank Mustansiriyah University (WWW.uomustansiriyah.edu.iq) Baghdad –Iraq for its support in the present work.

References

[1]Agarwal, R.P., Baleanu, D., Nieto, J.J., Torres, D.F., Zhou, Y. A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. J. Comput. Appl. Math.2018, 339, 3–29 .

[2] Ahmad, S., Ullah, A., Shah, K., Salahshour, S., Ahmadian, A., Ciano, T.: Fuzzy fractional-order model of the novel coronavirus. Adv. Differ. Equ.2020, 2020:472, 1–17 .

[3] Ahmadian, A., Ismail, F., Salahshour, S., Baleanu, D., Ghaemi, F. Uncertain viscoelastic models with fractional order: a new spectral tau method to study the numerical simulations of the solution. Commun. Nonlinear Sci. Numer. Simul.,2017, 53, 44–64 .

[4] Alderremy, A., Gómez-Aguilar, J., Aly, S., Saad, K.M. A fuzzy fractional model of coronavirus (COVID-19) and its study with Legendre spectral method. Results Phys.,2021, 21, 103773 .

[5] Balasubramaniam, P., Muthukumar, P., Ratnavelu, K. Theoretical and practical applications of fuzzy fractional integral sliding mode control for fractional-order dynamical system. Nonlinear Dyn. ,2015,80(1), 249–267 .

[6] Dong, N.P., Long, H.V., Giang, N.L. The fuzzy fractional SIQR model of computer virus propagation in wireless sensor network using Caputo Atangana-Baleanu derivatives. Fuzzy Sets Syst.2021, 429(3), 28–59 .

[7] Vu, H., Van Hoa, N. Hyers-Ulam stability of fuzzy fractional volterra integral equations with the kernel ψ-function via successive approximation method. Fuzzy Sets Syst,2020, 419, 67–98 .

[8] Wang, X., Luo, D., Zhu, Q. Ulam-Hyers stability of caputo type fuzzy fractional differential equations with time-delays. Chaos Solit. Fractals,2022 156(147), 111822 .

[9] Trujillo, J., Rivero M., Bonilla B. On a Riemann-Liouville generalized Taylor's formula J. Math. Anal. Appl., 1999,241 (1), 255-265.

[10] Anastassiou, G. Opial type inequalities involving Riemann-Liouville fractional derivatives of two functions with applications, Math. Comput. Model., 2008, 48 (3–4) , 344-374.

[11] Liang Y.S., Box dimensions of Riemann-Liouville fractional integrals of continuous functions of bounded variation Nonlinear Anal. Theory Methods Appl., 2010, 72 (11) , 4304-4306.

[12] Almeida R., D.F.M. Torres Calculus of variations with fractional derivatives and fractional integrals Appl. Math. Lett., 2009, 22 (12) , 1816-1820.

[13] Mohammad A. , Mohammed Al-Smadi , Rokiah R. Ahmad and Ummul K. Salma Din , An Analytical Numerical Method for Solving Fuzzy Fractional Volterra Integro-Differential Equations Volterra-Fredholm integro-differential equation. Symmetry,2019,11(2):205.

[14] Naveed A., Abd U., Aman U., Shabir A., Kamal S., On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation, Alexa.Eng. , 2020,60(1),1827-1838.

[15] Priyadharsini, S., Parthiban, V., Manivannan, A. Solution of fractional integro-differential system with fuzzy initial condition. Int. J. Pure Appl. Math. 2016,106(8), 107–112.

[16]Shabestari, M.R., Ezzati, R.,Allahviranloo, T.Numerical solution of fuzzy fractional integro-differential equation via two-dimensional Legendre wavelet method. J. Intell. Fuzzy Syst, 2018, 34(4), 2453–2465.

[17] Padmapriya, V., Kaliyappan, M., Parthiban, V. Solution of Fuzzy Fractional Integro-Differential Equations Using A Domian Decomposition Method. J. Inform. Math. Sci., 2017, 9(3), 501–507.

[18] Matinfar,M., Ghanbari,M., Nuraei, R.Numerical solution of linear fuzzy Volterra integro-differential equations by variational iteration method. J. Intell. Fuzzy Syst. 2013, 24(3), 575–586.

[19]Gumah, G., Moaddy, K., Al-Smadi, M., Hashim, I. Solutions to uncertain Volterra integral equations by fitted reproducing kernel Hilbert space method. J. Funct. Spaces 2016(50), 1-11.

[20]Zhang, H., Liao, X., Yu, J. Fuzzy modeling and synchronization of hyperchaotic systems. Chaos Solitons Fractals, 2005, 26(3), 835–843.

[21]El Naschie, M.S. From experimental quantum optics to quantum gravity via a fuzzy Kähler manifold. Chaos Solitons Fractals, 2005, 25(5), 969–977.

[22] Rasha H. Ibraheem , Rawaa I. Esa , Ali F. Jameel. The New Runge-Kutta Fehlberg Method for the Numerical Solution of Second-Order Fuzzy Initial Value Problems, Mathematical Modelling of Engineering Problems , 2023, 10(4), 1409-1418.

[23]Feng, G., Chen, G. Adaptive control of discrete time chaotic systems: a fuzzy control approach. Chaos Solitons Fractals , 2005, 23(2), 459–467.

[24] Nour K. Salman, Muna M. Mustafa. Numerical Solution of Fractional Volterra-Fredholm Integro-Differential Equation Using Lagrange Polynomials. Baghdad Sci. 2020, 17(4):1234-1240.

[25] Ting Deng, Jin Huang, Yifei Wang, Hu Li .A generalized fuzzy barycentric Lagrange interpolation method for solving two-dimensional fuzzy fractional Volterra integral equations. Numerical Algorithms.2024, 98(2):743–766.

[26] Moi, S., Biswas, S., Sarkar, S.P.: A Lagrange spectral collocation method for weakly singular fuzzy fractional Volterra integro-differential equations. Soft Comput. 2024, 27(8), 4483–4499 .

[27] Kaleva, O. Fuzzy differential equations. Fuzzy Sets Syst. 1987, 24, 301–317.

[28] Goetschel, J.R., Voxman, W. Elementary fuzzy calculus. Fuzzy Sets Syst. 1986, 18, 31–43.

[29] Puri, M.L., Ralescu, D.A. Differentials of fuzzy functions. J. Math. Anal. Appl. 1983, 91(2), 552–558.

[30] Bede, B., Gal, S.G. : Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. Fuzzy Sets Syst. 2005, 151(3), 581–599.

[31] Moaddy, K., Freihat, A.; Al-Smadi, M.; Abuteen, E.; Hashim, I. Numerical investigation for handling fractional-order Rabinovich–Fabrikant model using the multistep approach. Soft Comput. 2018, 22(3), 773–782.

[32] Chalco-Cano, Y., Roman-Flores, H. On new solutions of fuzzy differential equations. Chaos Solitons Fractals 2008, 38, 112–119.

[34] Salahshour, S., Allahviranloo, T., Abbasbandy, S.; Baleanu,D. Existence and uniqueness results for fractional differential equations with uncertainty. Adv. Differ. Equ. 2012, 1, 112.

[34] Mojahedfar M, Tari A. Solving two-dimensional fractional integro-differential equations by legendre wavelets. Bull. Iran. Math. Soc. , 2017, 43(7):2419–2435.

[35] Mazandarani, M., Kamyad, A.V. Modified fractional Euler method for solving fuzzy fractional initial value problem. Commun. Nonlinear Sci. and Numer. Simul. , 2013, 18(1), 12–21.