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| RESEARCH ARTICLE

Construction of Minimum Level Change, Linear Trend Free Fractional Factorial Designs using Projective Geometry

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| ABSTRACT
Globally minimum level change and linear trend free fractional factorial designs are constructed using the distinct points of Projective Geometry and the technique given by Chen and Wang (2001).

| KEYWORDS

Projective Geometry, Trend free designs, Minimum level Change design

| ARTICLE INFORMATION

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1. Introduction

One of the basic principal in designing an experiment is randomization. However in some situations randomization can have significant impact on the efforts and cost of the experimentation, as there may be factors that are difficult to change (e.g., changes of mould) or that require stabilization time to obtain adequate operating conditions (e.g., the temperature of an oven). Another situation where randomization may lead to undesirable run order is when the factor effects are aliased with the time trend. For example, if a batch of material is created at the beginning of an experiment and treatments are to be applied to experimental units formed from the material over time then there could be an unknown effect due to the aging of material which influences the observations obtained. Thus, it seems reasonable to consider systematic run order in which the factors with expensive or difficult to vary levels are minimally varied during an experimentation and the desired effects are orthogonal to the unknown trend.

The study of such systematic designs was initiated by Cox(1952), Daniel and Wilcoxon(1966) and further studied by Bailey(1983) and Cheng and Jacroux(1988). They did not consider the number of factor level changes. Coster and Cheng(1988) gave GFS to generate systematic run orders for systematic factorial plans. The minimum cost run order for fractional factorial experiments with all factors in s levels, where s is a prime is given by Chen and Wang (2001). They first constructed ordered s-level orthogonal arrays since all the main effects are estimable in an orthogonal array, and assigned factors to appropriate columns with a smaller number of level changes to get a design with global minimum cost run order using Generalised Foldover Scheme, given by Coster and Cheng(1988). Chen and Wang(2001) also mentioned the problem of obtaining appropriate defining contrasts for finding optimal run orders. They used a three step procedure to obtain a saturated ordered q-level orthogonal array and then assigned factors to the appropriate columns to obtain a design with the global minimum cost run order.

In this paper, we construct (i) globally minimum level change fractional factorial designs and (ii) linear trend free globally minimum level change fractional factorial designs, using the points in Projective Geometry and the assignment procedures given by Chen and Wang (2001). Once the designs are constructed, their defining relationships are also obtained. This overcomes the problem of obtaining appropriate defining contrasts for optimal run orders. The paper is organized as follows. Section 2 gives the preliminaries required. The construction technique for minimum level change design is given in Section 3. Section 4 gives the construction method for the designs satisfying the dual criteria of minimum level change and trend free property.

2. Preliminaries

Definition 1: Ordered Orthogonal Array

An orthogonal array $OA_N(q^r)$ is an N×r matrix containing r, q-level columns in which all possible combinations of levels in any two columns appear the same number of times, where $N = q^k$ for some integer k. If k = (N-1)/(q-1) then an array is called a saturated orthogonal array.

An orthogonal array is an ordered orthogonal array if its factors are assigned in increasing order of their level changes. In an ordered two level orthogonal array , the i^{th} column changes its symbols i times, either from 0 to 1 or from 1 to 0, where 0 and 1 denotes the two levels in the array.

Definition 2: Projective Geometry PG(R,q)

Let q be a prime or prime power, then GF(q) contains q elements say $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{q-1}$. Any ordered set of (R+1) elements $(x_0, x_1, x_2, \ldots, x_R)$ where $x_i's$ belong to GF(q) and are not all simultaneously zero is called a point in PG(R,q). The total number of points in PG(R,q) is $q^{R+1} - 1$. The elements $x_0, x_1, x_2, \ldots, x_R$ are called the coordinates of the point $(x_0, x_1, x_2, \ldots, x_R)$. Two sets $(x_0, x_1, x_2, \ldots, x_R)$ and $(y_0, y_1, y_2, \ldots, y_R)$ will represent the same point if $y_i = ?\ x_i$(i= 0,1,2,...,R)and ρ≠0ϵGF(q). The total number of distinct points in PG(R,q) is $Q_R = \frac{q^{R+1}-1}{q-1}$, since apart from 0 and 1 there are (q-1) possibilities for ρ. Thus, the geometry PG(R,q) has $\frac{q^{R+1}-1}{q-1}$ distinct points.

Definition 3: Generalised Foldover Scheme

Coster and Cheng (1988) introduced Generalised Foldover Scheme to generate systematic run order for factorial plans.

For a factorial design (q^k), $k = n-p$, Let U_0 be a 1×*n* matrix of zeros. Then the run order of design D produced by the GFS with respect to the generator sequence G= $(g_1, g_2, \ldots, g_k)'$ is given by U_k where

$$U_i = U_{i-1}^*(g_i) = \begin{bmatrix} U_{i-1} \\ U_{i-1}(g_i) \\ ? \\ U_{i-1}((q-1)g_i) \end{bmatrix}$$

for *i = 1,2,...,k*.

In the run order U_{n-p}, the principal block consists of the first q^{n-p-r} runs, the second block consists of the next q^{n-p-r} runs and so on.

Definition 4: Time Count

Let $Y = (y_1, y_2, \ldots y_N)'$ denote the ordered vector of observations, and $T_x = (1^x, 2^x, \ldots N^x)'$ for *x*= 0,1,2,...,*v* be an *N*×*1* vector of trend coefficients and let $\boldsymbol{a_i}$ be the contrast for main effect A_i, $i = 1,2,\ldots,n$ in the run order. Then the quantity $\boldsymbol{a_i' T_x}$ is known as the time count for main effect $\boldsymbol{A_i}$.

A necessary and sufficient condition for a main effect contrast *a* to be *v*-trend free is that

$$a'T_x = 0 \quad \forall\, x = 0,1,2,\dots,v \qquad (1)$$

When the assumption of trend effect of different factors of the design is also considered,

The model is

$$y = X\beta + B\gamma + T\theta + \varepsilon$$

Where y is the N × 1 column vector of observations , X is the N× n design model matrix of known constants; $\boldsymbol{\beta}$ is the n×1 column vector of regression coefficients and e is the N×1 column vector of random errors with zero means and variance σ^2 and , T = $(t)_{N\times 1}$; t is (b × 1) the linear trend vector and θ is the trend coefficient.

Definition 5: Optimal run order

A run order is optimal for the estimation of factor effects of interest in the presence of nuisance *v*- degree polynomial trend if

$$X'T = 0 \qquad \textbf{(2)}$$

A1. Minimum Level Change Design

Chen and Wang (2001) proposed the following assignment procedure to construct the global minimum level change designs.

***Assignment procedure I*: Resolution III q-level fractional factorial experiments**

A global minimum cost run order for a resolution III, q- level fractional factorial design with n factors and (N/q -1) / (q-1) +1≤ n ≤ (N-1) / (q-1), can be constructed by selecting the first n columns in the ordered $OA_N(q^{(N-1)/(q-1)})$. If t ≤ n ≤ (N/q -1)/(q-1), then the columns $\boldsymbol{X_1}$, (q-1)$\boldsymbol{X_1}$+ $\boldsymbol{X_2}$, (q-1)$\boldsymbol{X_2}$+$\boldsymbol{X_3}$,…,(q-1)$\boldsymbol{X_{k-1}}$+$\boldsymbol{X_k}$ are selected and from the remaining columns first n-t columns in the ordered orthogonal arrays are selected for a design with the global minimum cost run order.

***Assignment procedure II*: Two-level fractional factorial experiments with specified requirements**

Consider a two-level fractional factorial experiment with n factors $A_1, A_2, \dots, A_n$ and m(<n)2-way interactions involvingA_1, say $A_1A_2, A_1A_3, \dots, A_1A_{m+1}$. let N/2 ≤ n ≤ N-1-m. First assign $A_1, A_2, \dots, A_{m+1}$ to columns N/2, N/2 – 1,…, N/2 – m in the ordered two level orthogonal array. The would give columns N-1, N-2, , N-m to 2-way interactions. Then, assign the remaining factors to the remaining columns with smaller number of level changes. This would result in a minimum cost run order.

When t ≤ n ≤ N/2, then assign $\boldsymbol{A_1}$ to column $\boldsymbol{X_{t-1}}$+$\boldsymbol{X_t}$. To assign remaining factors select n-1 columns as $\boldsymbol{X_1}$, $\boldsymbol{X_1}$+$\boldsymbol{X_2}$,$\boldsymbol{X_2}$+$\boldsymbol{X_3}$,…,$\boldsymbol{X_{t-2}}$+$\boldsymbol{X_{t-1}}$. Then pick n-t more columns with smaller number of level changes from the remaining columns in the orthogonal array. After n-1 columns are selected, randomly assign factors $\boldsymbol{A_2}$,…,$\boldsymbol{A_n}$ to them. These n assignments give a design with the global minimum cost run order.

3. Construction of Minimum level change designs using Projective Geometry

We construct the minimum level change designs using distinct points of Projective Geometry PG(R,q) and the assignment procedures given by *Chen and Wang* (2001).

Let **k =** $\frac{q^{R+1}-1}{q-1}$ denote the total number of distinct points in PG(R,q). We first consider the case of construction of two- level fractional factorial design.

AII. Construction of two- level fractional factorial design

We use the following steps for construction of 2^{n-p} designs

Method of Construction)(I)

1. Consider the distinct points of PG(R,2) and these are
k = $2^{R+1} - 1$ in number.

2. Write these points of PG(R,2) column-wise to form a
Matrix Z = (z_{ij}) Then Z is of order t×k where t= R+1 and
k= $2^{R+1} - 1$.

3. Obtain the generator matrix G= $(g_1\ g_2 \dots \dots g_t)'$ by applying the following transformation (*Chen and Wang* (2001)) on the rows of Z

$g_1 = z_1,\quad g_i = z_i - z_{i-1}$ For $2 \le i \le t$ (3)

4. Construct a saturated ordered array A = $OA_N(2^k)$, where N= 2^t by applying GFS on the generator sequence $g_1, g_2, \dots g_t$ where g_i*(i=1,2,...t)* denote the i[th] row of the matrix G.

5. In an ordered two- level orthogonal array the i[th] column changes level symbol i times. Therefore the number of level changes (NLC) for the i[th] column X_i of A, an ordered orthogonal array, is $(NLC)_{Xi} = i$.

6. For specific values of n we have the following procedures:

Case (i): For n ≥ N/2, t=n-p, m ≤ N-1-n, a 2^{n-p} fractional factorial design can be constructed by considering m(< t-1) two way interactions involving single factor A_1, say, A_1A_2, ……A_1A_{m+1} and applying assignment procedure II discussed in Section 2.1. i.e, assigning A_1, A_2,………A_{m+1} to the columns N/2, N/2-1,…N/2-m in the array A, the two-way interactions to the columns N-1,N-2,.., N-M and the remaining columns in order of increasing number of level changes.

Case(ii): For n < N/2, we apply assignment procedure II as discussed in section 2.1 i.e. assigning A_1 to column $X_{t-1}+X_t$ and for the remaining factors, select n-1 columns as X_1, X_1+X_2, $X_2 + X_3$,…… $X_{t-2} + X_{t-1}$. Then, pick n-t more columns with smaller numbers of level changes from the remaining columns in the ordered orthogonal array. After n-1 columns are selected, we randomly assign factors A_2, A_3……A_n to them. These n assignments give a design with the global minimum cost run order.

7. Select the columns of G corresponding to the columns assigned to n factors and form the generator matrix G_α of the constructed design.

8. The defining contrasts of the constructed design can be obtained by finding the null space of G_α.

Thus, a global minimum 2^{n-p} fractional factorial design with minimum number of level changes can be obtained.

Following examples illustrate the method of construction.

Example 1: Let q=2, R=2. The total number of distinct points in PG(2,2) is k = $\frac{2^3-1}{2-1}$ = 7 . The points are given in the following matrix

$$Z = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The generator matrix G is constructed as follows

$g_1 = z_1, \quad g_i = z_i - z_{i-1}$ for $2 \le i \le 3$

$$G=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}. \quad (4)$$

Applying GFS on G we get the ordered orthogonal array $OA_8(2^7)$ which is shown in Table 1. The last row gives the number of level changes (***NLC***) for each factor are also given.

Table 1: Ordered orthogonal array $OA_8(2^7)$

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
	0	0	0	0	0	0	0
	0	0	0	1	1	1	1
	0	1	1	1	1	0	0
	0	1	1	0	0	1	1
	1	1	0	0	1	1	0
	1	1	0	1	0	0	1
	1	0	1	1	0	1	0
	1	0	1	0	1	0	1
NLC	**1**	**2**	**3**	**4**	**5**	**6**	**7**
TC	**16**	**0**	**8**	**0**	**0**	**0**	**4**

Some fractional factorial designs with minimum number of level changes (*NLC*) are constructed using the method

a) 2^{5-2} fractional factorial design

Consider two 2-factor interactions A_1A_2, A_1A_3 involving one common factor A_1 and remainder factors are A_4, A_5.

n=5, p=2, N=8 and m= 2.Here, let m=2 then

$\frac{N}{2} = \frac{8}{2} = 4 = 5 = N - 1 - m = 8 - 1 - 2$=5 is satisfied.

Using the method discussed in section 3.1 and assign A_1, A_2, A_3 to column numbers (X_4, X_3, X_2) in the ordered two level orthogonal array (Table 1) . Further assign column number (X_7 , X_6) to 2-way interactions A_1A_2, A_1A_3 and the remaining factors A_4 and A_5 to columns X_1 and X_5 respectively . The 2^{5-2} design obtained from the above assignment has a global minimum cost run order with (4+3+2+1+5) = 15 level changes.

Since columns X_4, X_3, X_2, X_1 *and* X_5 are selected from the ordered orthogonal array $OA_8(2^7)$ the generator matrix G_α is

obtained by selecting the corresponding columns from the matrix G given in (4) and the matrix G_α is

$$G_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad (5)$$

By finding the null space of G_{α}, the defining contrast for the constructed designs is given as I= $A_1A_4A_5$ = $A_1A_2A_3$ *is* obtained. We give here the list of designs constructed using the method.

Table 2: List of generated Designs

PG(R,q)	Ordered OA	Designs	Generator Matrix	Defining Relation	Level Changes
PG(2,2)	$OA_8(2^7)$	2^{5-2}	$\begin{bmatrix}10001\\11101\\00111\end{bmatrix}$	$I=A_1A_4A_5=A_1A_2A_3$	15
PG(3,2)	$OA_{16}(2^{15})$	2^{10-6}	$\begin{bmatrix}1000000011\\1111100011\\0001110101\\0011001111\end{bmatrix}$	$I=A_1A_2A_4A_{10}=A_1A_2A_3A_9=A_2A_4A_8=A_2A_3A_7$ $=A_3A_4A_6=A_2A_3A_4A_5$	55
PG(3,2)	$OA_{16}(2^{15})$	2^{8-4}	$\begin{bmatrix}10000001\\11111001\\00011010\\00110101\end{bmatrix}$	$I=A_3A_4A_8=A_2A_4A_7=A_2A_3A_6=$ $A_2A_3A_4A_5$	36
PG(3,2)	$OA_{16}(2^{15})$	2^{9-5}	$\begin{bmatrix}100000001\\111110001\\000111010\\001100111\end{bmatrix}$	$I=A_1A_2A_3A_9=A_3A_4A_8=A_2A_4A_7=A_2A_3A_6=A_2A_3A_4A_5$	45
PG(3,2)	$OA_{16}(2^{15})$	2^{6-2}	$\begin{bmatrix}0000001\\1000111\\0010100\\0101110\end{bmatrix}$	$I=A_1A_2A_5=A_1A_2A_3A_4$	30

b). Construction of q^{n-p} Fractional Factorial Design where q>2

We present the following method for the construction of the designs.

Method of Construction II

1. Consider PG(R,q) for q>2. Let k = $\frac{q^{R+1}-1}{q-1}$ denotes the number of distinct points in it.
2. Write distinct points of PG(R,q) columnwise to form a Matrix Z = (z_{ij}). Then Z is of order t×k where t =R+1 And k= $\frac{q^{R+1}-1}{q-1}$ is the number of distinct points in the geometry.
3. Obtain the generator matrix G= (g_1 …… g_t)'by applying the following transformation (*Chen and Wang* (2001)) on the rows of Z

   $g_1 = z_1$, $g_i = z_i - z_{i-1}$ for $2 \le i \le t$

4. Construct a saturated ordered orthogonal array A= $OA_N(q^k)$; N=q^t by a applying GFS on the generator sequence $g_1\ g_2 \cdots\cdots g_t$ where g_i is the i[th] row of G.

5. Count the number of level changes (*NLC*) for each column j labeled as X_j where $NLC(X_j) = (q-1) ?_{i=1}^{t} q^{t-i} I(g_{ij})$, where g_{ij} is the j[th] element in g_i and $I(g_{ij})$ is an indicator function as defined in lemma.

6.

Case (i): In order to construct a resolution III, q^{n-p}; n-p =t, *q>2* fractional factorial design, where $\frac{q^{t-1}-1}{q-1}+1 = n = \frac{q^{t}-1}{q-1}$, Apply assignment procedure I of *Chen and Wang*(2001) i.e. selecting the first n columns in an ordered orthogonal array OA_N (q^k) and the corresponding first n columns of G will give the generator matrix G_{α}.

Case (ii): For the case t ≤ n ≤ (N/q-1)/(q-1), we select columnsX_1,$(q-1)X_1+X_2$, $(q-1)X_2+X_1$,.........$(q-1)X_{t-1}+X_t$ and the first n-t columns from the remaining columns in the ordered array for a design with global minimum-cost run order.

7. The defining contrasts for the constructed design can be obtained by finding the null space of G_α.

Thus, a global minimum q^{n-p} fractional factorial design with minimum number of level changes can be obtained.

The above method is illustrated through following examples.

Example 3: Let q=3, R=2. The total number of distinct points in PG(2,3) is k= $\frac{3^3-1}{3-1} = 13$. The points are given in the following

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The generator matrix G is constructed as

$g_1 = z_1, \quad g_i = z_i - z_{i-1}$ for 2 ≤ i ≤ 3

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \quad (6)$$

Using GFS an ordered $OA_{27}(3^{13})$ is constructed which is given in following Table 3.

Table 3: Ordered $OA_{27}(3^{13})$

	X₁	***X₂***	***X₃***	***X₄***	***X₅***	***X₆***	***X₇***	***X₈***	***X₉***	***X₁₀***	***X₁₁***	***X₁₂***	***X₁₃***
	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	1	2	1	2	2	1	2	1
	0	0	0	0	2	2	1	2	1	1	2	1	2
	0	2	2	1	0	0	0	1	1	1	1	2	2
	0	2	2	1	1	1	2	2	0	0	2	1	0
	0	0	0	1	0	0	1	1	0	0	1	1	0
	0	1	1	2	0	0	0	2	2	2	2	1	1
	0	1	1	2	1	1	2	0	1	1	0	0	2
	0	1	1	2	2	2	1	1	0	0	1	2	0
	2	0	1	1	0	1	1	0	0	1	2	2	1
	2	0	1	1	1	2	0	1	2	0	0	1	2
	2	0	1	1	2	0	2	2	1	2	1	0	0
	2	2	0	2	0	1	1	1	1	2	0	1	0
	2	2	0	2	1	2	0	2	0	1	1	0	1
	2	0	1	2	0	1	2	1	0	1	0	0	1
	2	1	2	0	0	1	1	2	2	0	1	0	2
	2	1	2	0	1	2	0	0	1	2	2	2	0
	2	1	2	0	2	0	2	1	0	1	0	1	1
	1	0	2	2	0	2	2	0	0	2	1	1	2
	1	0	2	2	1	0	1	1	2	1	2	0	0
	1	0	2	2	2	1	0	2	1	0	0	2	1
	1	2	1	0	0	2	2	1	1	0	2	0	1

	1	2	1	0	1	0	1	2	0	2	0	2	2
	1	0	2	0	0	2	0	1	0	2	2	2	2
	1	1	0	1	0	2	2	2	2	1	0	2	0
	1	1	0	1	1	0	1	0	1	0	1	1	1
	1	1	0	1	2	1	0	1	0	2	2	0	2
NL	2	6	8	8	1	2	2	2	2	26	26	26	26
TC	8	2	0	0	1	0	0	0	0	0	0	0	0

Now using G matrix and Table 3 we construct following 3-level fractional factorial designs.

b) 3^{8-5} Resolution III Fractional Factorial Design

Here $N=27=3^3$, q=3, t = 3.(N/q -1) / (q-1) +1 = 5 ≤ n ≤ 13 = (N-1) / (q-1). According to the step 6 of the method we select the first n = 8 columns in the ordered orthogonal array $OA_{27}(3^{13})$ given in Table 4. The generator matrix for the design is given by selecting the assigned columns from matrix G given in (6)

$$G_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Hence using the generator matrix, the defining relation is given $I=A_2A_5^2A_8 = A_1A_5A_7 = A_1A_5^2A_6 = A_1A_2A_4 = A_1A_2^2A_3$

Hence we get a global minimum cost 3^{8-5} ff design with 106 level changes, which is minimum possible. The list of constructed designs along with their defining relations and level changes is given in Table (4)

4. Minimum Level Change Trend free designs

Using the ordered orthogonal arrays obtained in Section 3, we can construct trend free designs. Another construction method for constructing q^{n-p} fractional factorial design satisfying the dual criteria of minimum number of level changes and linear trend free effects is given below:

Method of construction II:

1. We Construct ordered orthogonal array using GFS on the distinct points of PG(R,q) as described in section 3.
2. Calculate Time Count (**T.C**) for each factor in the ordered orthogonal array.
3. Choose first n columns labeled as $X_1, X_2, X_3, \ldots X_n$, from an ordered orthogonal array for which the time count is zero. Assign factors $A_1, A_2, \ldots, A_n$ to them.
4. Obtain the generator matrix G_α for q^{n-p} fractional factorial design by selecting the columns that correspond to the column assigned to the n factors $A_1, A_2, \ldots, A_n$ in an ordered orthogonal array.
5. Find the null space of G_α to obtain the corresponding defining relation.

We obtain a resolution III, q^{n-p} fractional factorial design with minimum number of level changes and with linear trend free main effects. Consider the following examples

a) 2^{5-1} Resolution III Fractional Factorial Design

Select the columns in Ordered Orthogonal Array (Table 2) with zero time counts. The selected columns are X_2, X_4, X_5, X_6, X_8 and the corresponding generator matrix is given as

$$G_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This gives a 2^{5-1} fractional factorial design with defining relation $I = A_1A_2A_4$, with 25 level changes (minimum possible satisfying the dual criteria) in which all main effects are linear trend free.

b) 3^{5-2} Resolution III Fractional Factorial Design

Here N=27, q=3, n=5, p=2.(N/q -1) / (q-1) +1 = 5 ≤ 5 ≤ 13 = (N-1) / (q-1). According to the assignment I, select the first n = 5 columns in the ordered orthogonal array $OA_{27}(3^{13})$ given in Table 3. The generator matrix for the design is given by selecting the assigned columns from matrix G given in (6).

$$G_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Hence using the generator matrix, the defining relation is given as $I=A_1A_2A_4 = A_1A_2^2A_3$.Hence we get a global minimum cost design with 42 level changes.

Table 5 and 6 gives possible 2^{n-p}, $5 \le n \le 11$, $1 \le p \le 7$ and 3^{n-p}, $4 \le n \le 10$, $1 \le p \le 7$ resolution III fractional factorial designs respectively satisfying the dual criteria minimum level change and linear trend free. The last column of the table gives the defining relations for the designs.

Table 4: List of generated Designs with their level changes

PG(R,q)	Ordered OA	Designs	Generator Matrix	Def. Relation	Level Changes
PG(2,3)	$OA_{27}(3^{13})$	3^{8-5}	$\begin{bmatrix} 00001121 \\ 02210001 \\ 20110110 \end{bmatrix}$	$I=A_2A_5^2A_8 = A_1A_5A_7$ $A_1A_5^2A_6=A_1A_2A_4=$ $A_1A_2^2A_3$	106
PG(2,3)	$OA_{27}(3^{13})$	3^{7-4}	$\begin{bmatrix} 0000112 \\ 0221000 \\ 2011011 \end{bmatrix}$	$I=A_1A_5^2A_6$ $=A_1A_5A_7= A_1A_2^2A_3$ $A_1A_4A_2$	82
PG(2,3)	$OA_{27}(3^{13})$	3^{9-6}	$\begin{bmatrix} 000011212 \\ 022100011 \\ 201101100 \end{bmatrix}$	$I=A_2A_5^2A_8=A_1A_5A_7=$ $A_2A_1A_4=A_1A_5^2A_6$ $A_1A_2^2A_3$	130
PG(2,3)	$OA_{27}(3^{13})$	3^{5-2}	$\begin{bmatrix} 00001 \\ 02210 \\ 20110 \end{bmatrix}$	$I=A_1A_2A_4=$ $A_1A_2^2A_3$	42
PG(2,3)	$OA_{27}(3^{13})$	3^{10-7}	$\begin{bmatrix} 0000112122 \\ 0221000111 \\ 2011011001 \end{bmatrix}$	$I=A_2A_5^2A_8=A_1A_5A_7$ $A_1A_5^2A_6=A_1A_2A_4=$ $A_1A_2^2A_3= A_1A_2A_5A_{10}$	156

Table 5: 2^{n-p} , 5 ≤ n ≤ 11, 1 ≤ p≤ 7 resolution III fractional factorial designs

Design	Col.Numselected in an OA(with tc=0)	Generator Matrix	NLC	Defining Relation
2^{5-1}	X_2, X_4, X_5, X_6, X_8	$\begin{bmatrix} 0&0&0&0&1 \\ 0&1&1&1&1 \\ 1&1&1&0&0 \\ 1&0&1&1&0 \end{bmatrix}$	25	$I = A_1A_2A_4$
2^{6-2}	$X_2, X_4, X_5, X_6, X_8\ X_9$	$\begin{bmatrix} 0&0&0&0&1&1 \\ 0&1&1&1&1&1 \\ 1&1&1&0&0&0 \\ 1&0&1&1&0&1 \end{bmatrix}$	34	$I = A_1A_2A_4 = A_2A_3A_5A_6$
2^{7-3}	$X_2, X_4, X_5, X_6, X_8\ X_9, X_{10}$	$\begin{bmatrix} 0&0&0&0&1&1&1 \\ 0&1&1&1&1&1&1 \\ 1&1&1&0&0&0&1 \\ 1&0&1&1&0&1&1 \end{bmatrix}$	44	$I = A_1A_2A_4 = A_2A_3A_5A_6 = A_1A_5A_7$
2^{8-4}	$X_2, X_4, X_5, X_6, X_8\ X_9, X_{10}, X_{11}$	$\begin{bmatrix} 0&0&0&0&1&1&1&1 \\ 0&1&1&1&1&1&1&1 \\ 1&1&1&0&0&0&1&1 \\ 1&0&1&1&0&1&1&0 \end{bmatrix}$	55	$I = A_1A_2A_4 = A_2A_3A_5A_6$ $= A_1A_5A_7 = A_1A_2A_3A_5A_8$
2^{9-5}	$X_2, X_4, X_5, X_6, X_8\ X_9, X_{10}, X_{11}, X_{12}$	$\begin{bmatrix} 0&0&0&0&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&0 \\ 1&1&1&0&0&0&1&1&1 \\ 1&0&1&1&0&1&1&0&0 \end{bmatrix}$	67	$I = A_1A_2A_4 = A_2A_3A_5A_6$ $= A_1A_5A_7$ $= A_1A_2A_3A_5A_8 = A_2A_5A_9$
2^{10-6}	$X_2, X_4, X_5, X_6, X_8\ X_9, X_{10}, X_{11}, X_{12}, X_{13}$	$\begin{bmatrix} 0&0&0&0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&0&0 \\ 1&1&1&0&0&0&1&1&1&1 \\ 1&0&1&1&0&1&1&0&0&1 \end{bmatrix}$	80	$I = A_1A_2A_4 = A_2A_3A_5A_6 = A_1A_5A_7 =$ $A_1A_2A_3A_5A_8 = A_2A_5A_9 = A_3A_5A_{10}$
2^{11-7}	$X_2, X_4, X_5, X_6, X_8\ X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}$	$\begin{bmatrix} 0&0&0&0&1&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&0&0&0&1&1&1&1&0 \\ 1&0&1&1&0&1&1&0&0&1&1 \end{bmatrix}$	94	$I = A_1A_2A_4 = A_2A_3A_5A_6$ $= A_1A_5A_7 = A_1A_2A_3A_5A_8$ $= A_2A_5A_9 = A_3A_5A_{10} = A_1A_2A_5A_{11}$

Table 6: 3^{n-p} , 4 ≤ n ≤ 10, 1 ≤ p ≤ 7 resolution III fractional factorial designs

Design	Columns selected (with time count zero)	Generator matrix	NLC	Defining Relation
3^{4-1}	X_3, X_4, X_6, X_7	$\begin{bmatrix} 0&0&1&2 \\ 2&1&0&0 \\ 1&1&1&1 \end{bmatrix}$	56	$I = A_1^2A_2^2A_3A_4$
3^{5-2}	X_3, X_4, X_6, X_7, X_8	$\begin{bmatrix} 0&0&1&2&1 \\ 2&1&0&0&1 \\ 1&1&1&1&0 \end{bmatrix}$	80	$I = A_1A_3^2A_5 = A_1^2A_2^2A_3A_4$
3^{6-3}	$X_3, X_4, X_6, X_7, X_8, X_9$	$\begin{bmatrix} 0&0&1&2&1&2 \\ 2&1&0&0&1&1 \\ 1&1&1&1&0&0 \end{bmatrix}$	104	$I = A_2^2A_3A_6 = A_1A_3^2A_5 = A_1^2A_2^2A_3A_4$
3^{7-4}	$X_3, X_4, X_6, X_7, X_8, X_9,$ X_{10}	$\begin{bmatrix} 0&0&1&2&1&2&2 \\ 2&1&0&0&1&1&1 \\ 1&1&1&1&0&0&1 \end{bmatrix}$	130	$0I = A_1A_3A_7 = A_2^2A_3A_6 = A_1A_3^2A_5 = A_1^2A_2^2A_3A_4$
3^{8-5}	$X_3, X_4, X_6, X_7,$ $X_8, X_9,$ X_{10}, X_{11}	$\begin{bmatrix} 0&0&1&2&1&2&2&2 \\ 2&1&0&0&1&1&1&2 \\ 1&1&1&1&0&0&1&2 \end{bmatrix}$	156	$I = A_2^2A_4^2A_8 = A_1A_3A_7 = A_2^2A_3A_6 = A_1A_3^2A_5 = A_1^2A_2^2A_3A_4$
3^{9-6}	$X_3, X_4, X_6, X_7, X_8, X_9,$ X_{10}, X_{11}, X_{12}	$\begin{bmatrix} 0&0&1&2&1&2&2&1&2 \\ 2&1&0&0&1&1&1&1&2 \\ 1&1&1&1&0&0&1&2&2 \end{bmatrix}$	182	$I = A_1A_2^2A_3A_9 = A_2^2A_4^2A_8 = A_1A_3A_7 = A_2^2A_3A_6 = A_1A_3^2A_5 = A_1^2A_2^2A_3A_4$

3^{10-7}	X_3, X_4, X_6, X_7,X_8,X_9, X_{10}, X_{11},X_{12},X_{13}	$\begin{bmatrix}0&0&1&2&1&2&2&1&2&1\\2&1&0&0&1&1&1&1&2&2\\1&1&1&1&0&0&1&2&2&1\end{bmatrix}$	208	$I = A_1A_2^2A_3^2A_{10} = A_1A_2^2A_3A_9 = A_2^2A_4^2A_8 = A_1A_3A_7 = A_2^2A_3A_6 = A_1A_3^2A_5 = A_1^2A_2^2A_3A_4$

5. Conclusion

This article provides a streamlined methodology for constructing globally optimal level change designs utilizing distinct points from projective geometry. Furthermore, this approach is employed to devise minimum level change fractional factorial designs that adhere to the dual principles of trend freeness and the minimization of level changes, along with their corresponding defining relations.

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