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| RESEARCH ARTICLE

An Efficient Estimator to Estimate the Population Mean for a Sensitive Variable Using Dual Auxiliary Information

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| ABSTRACT
Randomized response technique (RRT) is designed to mitigate bias resulting from evasive responses in surveys. This article introduces a class of difference type RRT estimator for estimating finite population mean of a sensitive characteristic, leveraging information on auxiliary variables, as well as their ranks. The expressions of Bias and MSE are derived up to first order approximation for the proposed estimator. Empirical comparisons are made which demonstrate the superiority of our approach, particularly when incorporating information on the ranks of auxiliary variables. Our findings indicate that our proposed estimator outperforms existing alternatives developed for similar scenarios.

| KEYWORDS

Auxiliary information, Sensitive variable, Bias, Mean squared error

| ARTICLE INFORMATION

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1. Introduction

In the survey, the researcher faces many problems when the primary data is sensitive. To collect information on sensitive and stigmatizing characteristics from the human populations is not easy matter. That is gathering information the interviewee hesitates to respond their private life, as their sexual orientation, experience in induced abortion, drug addiction, HIV infection status, duration of suffering from AIDS, the incidence of domestic violence, maltreatment of spouse, and tax evasion, for example, any kinds of sensitive query etc., people provide false information intentionally, we may not obtain trustworthy information easily by the respondents. Warner (1965) gave an ingenious method to estimate the incidence of attributes of sensitive characters to reduce the bias on evasive response, as a RRT (model). Several other researchers have extended this work on randomized response models with scrambling responses for estimation of population mean in finite sampling which includes Pollock and Bek (1976), Echhorn and Hayre (1983), Mangat and Singh (1990), Singh *et al.* (2000), Gupta *et al.* (2002), Gupta and Shabbir (2004), Gupta *et al*. (2010). They presented the theory proposing additive, multiplicative scrambling randomized response models without having auxiliary information. In the last decades researchers extended results related to RRTs models utilizing auxiliary information for the mean estimation of sensitive variable. Sousa *et al*. (2010) firstly presented ratio-type estimators for estimating population mean using auxiliary information. Diana and Perri's (2011) presented RRT model using two scrambling variables. Koyuncu, K. (2012) proposed efficient estimators of population mean using auxiliary attributes. Koyuncu *et al.* (2014) presented exponential-type estimators using one and two auxiliary variables. Gupta et al. (2012), Gupta *et al*. (2014 and 2015) presented the additive ratio and regression estimator using RRT and ORRT models. Gupta *et al*. (2016), Mushtaq *et al*. (2017), Noor-ul-Amin *et al*. (2018), Khalid *et al*. (2018 and 2019), Waseem *et al*. (2020) used auxiliary attributes in RRT and ORRT models. Sanaullah *et al*. (2020), Saleem and Sanaullah (2022), Tiwari *et al*. (2022), Khalid *et al*. (2023) presented various estimators for the population mean of the complex variable using different sampling designs under different RRT and ORRTs models. Saleem *et al*. (2023) proposed a generalized estimator for variance estimation of sensitive variable used of two auxiliary attributes in RRT models and Azeem *et al*. (2024) proposed a new RRT estimator for variance estimation. It has been shown that the use of non-sensitive auxiliary information is very suitable in increasing the efficiency of the estimators.

The present article suggest a difference type exponential improved estimator of population mean of sensitive variable of interest using RRT models motivated by the work of Haq *et al*.(2016), and Tiwari *et al*. (2022). The proposed estimator is defined by utilizing the information on the auxiliary variable and information on the ranks of the auxiliary variable. The mathematical expressions of the bias and the mean square error MSE of the proposed estimator are derived up to first order of approximation. Numerical illustrations show that the proposed estimator performs better than all estimators considered here.

The layout of rest of the article organized as follows. In section 2 we discussed the terminology for the scrambled response RRT model. In section 3, we listed some existing estimators of the population mean for the scrambled response using additive model. In section 4 a class of difference type RRT estimator is proposed for estimation of population mean of the sensitive variable for finite population and obtained the expressions of bias and MSE. In section 5, we evaluate efficiency comparison of the proposed and existing estimators. In section 6 we discussed the simulation study to support the empirical findings. In section 7 interpretations of the results, and Section 8 final concludes of the study.

2. Terminology

In a finite sample survey let the characteristics under study Y be a sensitive variable which cannot be observed directly and X be a non-sensitive auxiliary variable which is correlated with $Y$and the ranks of the auxiliary variable be denoted by R_x. Using the additive model of Pollock and Bek (1976) we observe the scrambled response variable defined as $Z = Y + S$, where S is scrambled response variable independent of Y and X with the assumption that S has a known distribution with mean zero and variance S_s^2, i.e. $E\,(Z) = E\,(Y) + E(S)$, as $E(S) = 0$, $E\,(Z) = E(Y)$, i.e. $\bar{Z} = \bar{Y}$. The respondent is asked to report a scrambled response for Y given by $Z = Y + S$ and is asked to provide a true response for X.

Suppose that a simple random sample without replacement of size n be drawn from a finite population $U = (U_1, U_2 \cdots, U_N)$, for the i^{th} observed unit $(i = 1,2,\cdots\cdots,N)$, let z_i, x_i and r_i respectively, be the values of the scramble variable Z, auxiliary variables X and ranks of the auxiliary variable R_x. Let $\bar{y} = \sum_i^n y_i/n$, $\bar{x} = \sum_i^n x_i/n$ and $\bar{r}_x = \sum_i^n r_i/n$ be the sample means y and x and r_x respectively and $\bar{X} = \frac{1}{N}\sum_{i=1}^N X_i = E(X)$, $\bar{Y} = \frac{1}{N}\sum_{i=1}^N Y_i = E\,(Y)$, $\bar{Z} = \frac{1}{N}\sum_{i=1}^N Z_i = E(Z)$, be the corresponding population mean of X, Y, Z respectively, $\bar{R}_x = \frac{1}{N}\sum_{i=1}^N R_{x_i}$ is the population mean of R_x, $S_z^2 = \frac{1}{N-1}\sum_{i=1}^N (Z_i - \bar{Z})^2$ $S_y^2 = \frac{1}{N-1}\sum_{i=1}^N (Y_i - \bar{Y})^2$, $S_s^2 = \frac{1}{N-1}\sum_{i=1}^N (S_i - \bar{S})^2$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^N (X_i - \bar{X})^2$ and $S_{r_x}^2 = \frac{1}{N-1}\sum_{i=1}^N \left(R_{x_i} - \bar{R}_x\right)^2$ are the population variances, of Y, X, and Z, S and R_x, respectively. The assumption is that $\bar{X}$ is known and $\bar{S} = \mathrm{E}\,(\mathrm{S}) = 0$, thus $E(Z) = E(Y)$ and $C_z^2 = C_y^2 + \left(\frac{S_s^2}{\bar{Y}^2}\right)$, C_y, C_x, C_z, and C_r are the coefficients of variation of Y, X, Z and R_x respectively.

3. Existing Estimators of Population Mean using RRT additive model

For estimation of population mean various other estimators are proposed under additive model, $Z = Y + S$. These estimators and their $Bias$ and MSE up to first order of approximation, are given as follows:

(i) RRT Mean Estimator suggested by Sousa *et al*. (2010) is

$$t_Y = \frac{1}{n}\sum_{i=1}^{n} z_i = \bar{z} \tag{1}$$

Variance of t_Y is given by

$$t_Y = \lambda(S_z^2 + S_s^2) \quad (2)$$

(ii) RRT Ratio estimator suggested by Sousa *et al*. (2010)

$$t_R = \bar{z}\left(\frac{\bar{X}}{\bar{x}}\right) \quad (3)$$

The Bias and MSE of t_R to the first degree of approximation is given by

$$Bias(t_R) \cong \lambda\bar{Y} (C_x^2 - \rho_{zx}C_zC_x) \quad (4)$$

$$MSE(t_R) \cong \bar{Y}^2\lambda(C_z^2 + C_x^2 - 2\rho_{zx}C_zC_x) \quad (5)$$

(iii) RRT Regression type Estimator suggested by Gupta *et al*. (2012)

$$t_{Reg} = \bar{z} + \hat{\beta}_{zx}(\bar{X} - \bar{x}) \quad (6)$$

$$Bias\left(t_{Reg}\right) \cong -\beta_{zx}\,\lambda\left\{\frac{\mu_{12}}{\mu_{11}} - \frac{\mu_{03}}{\mu_{02}}\right\} \quad (7)$$

$$MSE\left(t_{Reg}\right) \cong \bar{Y}^2\lambda C_z^2(1 - \rho_{zx}^2)^2 \quad (8)$$

(iv) RRT Generalized Regression-cum-ratio Estimator, suggested by Gupta *et al*. (2012)

$$t_{GRR} = [k_1\bar{z} + k_2(\bar{X} - \bar{x})]\left(\frac{\bar{X}}{\bar{x}}\right) \quad (9)$$

$$Bias(t_{GRR}) = [(k_1 - 1)\bar{Y} + k_1\bar{Y}\lambda\{C_z^2 - \rho_{zx}C_zC_x\} + k_2\bar{X}\lambda C_x^2] \quad (10)$$

$$MSE_{min}(\mathrm{t}_{GRR}) \cong \bar{Y}^2\left[\frac{C_z^2(1-\rho_{zx}^2)\lambda\{1 - \lambda C_x^2\}}{C_z^2(1-\rho_{zx}^2)\lambda + \{1 - \lambda C_x^2\}}\right] \quad (11)$$

Where, $and\ k_{1(opt)} = \left\{\frac{1-\lambda\, c_x^2}{1-\lambda[C_x^2 - C_z^2(1-\rho_{zx}^2)]}\right\}, k_{2(opt)} = \frac{\bar{Y}}{\bar{X}}\left[1 + k_{1(opt)}\left(\frac{\rho_{zx}C_z}{C_x} - 2\right)\right]$

(v) RRT Generalized Exponential Estimator suggested by Koyuncu *et al*. (2014)

$$t_{GER} = [w_1\bar{z} + w_2(\bar{X} - \bar{x})]\exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \quad (12)$$

$$Bias(t_{GER}) \approx (w_1 - 1)\bar{Y} + \lambda\left[\frac{1}{2}w_1\bar{Z}\left(\frac{3}{4}C_x^2 - C_{zx}\right) + \frac{1}{2}w_2\bar{X}C_x^2\right] \quad (13)$$

$$MSE_{min}(\mathrm{t}_{GER}) \cong \bar{Y}^2\left[\left(1 - \frac{1}{4}\lambda C_x^2\right) - \frac{\left(\left(1 - \frac{1}{8}\lambda C_x^2\right)\right)^2}{1 + C_z^2(1-\rho_{zx}^2)}\right] \quad (14)$$

Where, $w_{1(opt)} = \left\{\frac{1-\frac{1}{8}\lambda C_x^2}{1+\lambda C_z^2(1-\rho_{zx}^2)}\right\}$ and $w_{2(opt)} = \frac{\bar{Y}}{\bar{X}}\left[\frac{1}{2} - w_{1(opt)}(1 - \rho_{zx}\frac{C_z}{C_x})\right]$

(vi) RRT Exponential ratio type estimator suggested by Gupta *et al*. (2017)

$$t_{ER} = \bar{z}\,exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \quad (15)$$

$$Bias(t_{ER}) \approx \lambda\bar{Y}\frac{1}{2}\left(\frac{3}{4}C_x^2 - \rho_{zx}\,C_zC_x\right) \quad (16)$$

$$MSE_{min}(\mathrm{t}_{ER}) \cong \bar{Y}^2\frac{1}{4}\lambda[(4C_z^2 - 4\rho_{zx}C_zC_x + C_x^2)] \quad (17)$$

(vii) By combining the regression, ratio and exponential estimators and further generalized mixture estimator given by:

$$\hat{\mu}_{GR} = \left\{d_1\bar{z}\left(\frac{\bar{X}}{\bar{x}}\right)^{\alpha} + d_2(\bar{X} - \bar{x})\right\}exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) \quad (18)$$

$$Bias(\hat{\mu}_{GR}) \approx (d_1 - 1)\bar{Y} + \lambda d_1\bar{Y}(BC_x^2 - A\rho_{zx}\,C_zC_x) + \lambda d_2\bar{X}\frac{1}{2}C_x^2 \quad (19)$$

$$MSE(\mathrm{t}_{GR}) \cong \bar{Y}^2\left\{\left(1 - \frac{1}{4}\lambda C_x^2\right) - \frac{\left(1+\lambda\left\{\left(\mathrm{B}-\frac{1}{2\mathrm{A}}-\frac{1}{4}\right)C_x^2 + \left(\frac{1}{2}-A\right)\rho_{zx}C_xC_z\right\}\right)^2}{1+\lambda\left[\left(2B-A-\frac{1}{4}\right)C_x^2 + (1-2A)\rho_{zx}C_zC_x + \left(1-\rho_{zx}^2\right)C_z^2\right]}\right\}$$

Where, $A = \alpha + \frac{1}{2}, B = \frac{1}{2}\alpha(\alpha+2) + \frac{3}{8}$ and $d_{1(opt)} = \frac{\left(1+\lambda\left\{\left(\mathrm{B}-\frac{1}{2\mathrm{A}}-\frac{1}{4}\right)C_x^2+\left(\frac{1}{2}-A\right)\rho_{zx}C_xC_z\right\}\right)^2}{1+\lambda\left[\left(2B-A-\frac{1}{4}\right)C_x^2+(1-2A)\rho_{zx}C_zC_x+\left(1-\rho_{zx}^2\right)C_z^2\right]}, d_{2(opt)} = \frac{\bar{Y}}{\bar{X}}\left\{\frac{1}{2} - d_{1(opt)}\left[\left(A+\frac{1}{2}\right) - \rho_{zx}\frac{C_z}{C_x}\right]\right\}$

$$MSE_{min}(\mathrm{t}_{GR}) \cong \bar{Y}^2\left\{\left(1-\frac{1}{4}\lambda C_x^2\right) - \frac{\left(1+\lambda\left\{\left(\mathrm{B}-\frac{1}{2\mathrm{A}}-\frac{1}{4}\right)C_x^2+\left(\frac{1}{2}-A\right)\rho_{zx}C_xC_z\right\}\right)^2}{1+\lambda\left[\left(2B-A-\frac{1}{4}\right)C_x^2+(1-2A)\rho_{zx}C_zC_x+\left(1-\rho_{zx}^2\right)C_z^2\right]}\right\} \qquad (20)$$

4. Proposed estimator

Motivated by Haq *et al*. (2017) Based on Tiwari *et al*. (2022) and we propose a class of difference type randomized response estimator of the population mean of a sensitive study variable Y, based on dual use of non-sensitive auxiliary variables i.e. using the information on the auxiliary variable X, and information on ranks of the auxiliary variable R_x, which is sufficiently correlated with the study variable Y. The proposed class of estimator is defined as

$$t_v = [w_1\bar{z} + w_2(\bar{X}-\bar{x}) + w_3(\bar{R}_x - \bar{r}_x)]\exp\left(\frac{\eta(\bar{X}-\bar{x})}{\bar{X}+\bar{x}}\right) \qquad (21)$$

Where w_1, w_2 and w_3 are the constants, whose values are to be determined by minimizing MSE(t_v) and η is suitable constants which takes real values.

4.1. The Bias and MSE of t_v

To get the expression of Bias and mean squared error (MSE) of the proposed class of estimator t_v . We write

$$e_0 = \left(\frac{\bar{z}-\bar{Z}}{\bar{Z}}\right), e_1 = \left(\frac{\bar{x}-\bar{X}}{\bar{X}}\right), e_2 = \left(\frac{\bar{r}_x - \bar{R}_x}{\bar{R}_x}\right)$$

Such that $E(e_0) = E(e_1) = E(e_2) = 0,\ E(e_0^2) = \lambda C_z^2,\ E(e_1^2) = \lambda C_x^2,\ E(e_2^2) = \lambda C_r^2$

$E(e_0e_1) = \lambda\rho_{zx}C_zC_x,\quad E(e_0e_2) = \lambda\rho_{zr_x}C_zC_r,\quad E(e_1e_2) = \lambda\rho_{xr_x}C_xC_r$

Where, $\lambda = \left(\frac{1}{n} - \frac{1}{N}\right),\ C_x^2 = \frac{S_X^2}{\bar{X}},\ C_Z^2 = \frac{S_Z^2}{\bar{Z}},\ C_{r_x}^2 = \frac{S_{r_x}^2}{\bar{R}_x}$

Expanding t_v in terms of $e's$ up to the first order of approximation, we have

$$t_v = \left[w_1\bar{Z}(1+e_0) + w_2\left(\bar{X}-\bar{X}(1+e_1)\right) + w_3\left(\bar{R}_x - \bar{R}_x(1+e_2)\right)\right]\exp\left(\frac{\eta\left(\bar{X}-\bar{X}(1+e_1)\right)}{\bar{X}+\bar{X}(1+e_1)}\right)$$

$$E(t_v - \bar{Y}) = \bar{Y}\left[w_1E\left\{1+e_0-\frac{\eta e_1}{2}-\frac{\eta e_0e_1}{2}+\frac{\eta e_1^2}{4}+\frac{\eta^2e_1^2}{8}\right\} - w_2\frac{1}{R}E\left\{e_1-\frac{\eta e_1^2}{2}\right\} - w_3\frac{1}{R^*}E\left\{e_2-\frac{\eta e_1e_2}{2}\right\} - 1\right]$$

$$Bias(t_v) = \bar{Y}\left[w_1\left\{1+\frac{\lambda\eta C_x^2}{4}\left(1+\frac{\eta}{2}-2\rho_{zx}\frac{C_z}{C_x}\right)\right\} + w_2\frac{1}{R}\left\{\frac{\lambda\eta C_x^2}{2}\right\} + w_3\frac{1}{R^*}\left\{\frac{\lambda\eta\rho_{xr_x}C_rC_x}{2}\right\} - 1\right] \qquad (22)$$

$$MSE(t_v) = \bar{Y}^2\left[1-2w_1\left\{1+\frac{\eta(\eta+2)\lambda C_x^2}{8}-\frac{\eta\lambda\rho_{zx}C_zC_x}{2}\right\} + w_1^2\left\{1+\lambda C_z^2+\frac{\eta(\eta+2)\lambda C_x^2}{4}-2\eta\lambda\rho_{zx}C_zC_x\right\} + w_2^2\frac{1}{R^2}\lambda C_x^2 + w_3^2\frac{1}{R^{*2}}\lambda C_r^2 - 2w_2\frac{1}{R}\frac{\eta\lambda C_x^2}{2} - 2w_3\frac{1}{R^*}\frac{\eta\lambda\rho_{xr_x}C_xC_r}{2} + 2w_1w_2\frac{1}{R}\lambda\{\eta C_x^2 - \rho_{zx}C_zC_x\} + 2w_1w_3\frac{1}{R^*}\lambda\{\eta\rho_{xr_x}C_xC_r - \rho_{zr_x}C_zC_r\} + 2w_2w_3\frac{1}{R}\frac{1}{R^*}\frac{\lambda\rho_{xr_x}C_xC_r}{2}\right]$$

The expression MSE of t_v is given as

$$MSE(t_v) = \bar{Y}^2[1-2w_1C_7 + w_1^2C_1 + w_2^2C_2 + w_3^2C_3 - 2w_2C_8 - 2w_3C_9 + 2w_1w_2C_4 + 2w_1w_3C_5 + 2w_2w_3C_6] \qquad (23)$$

Where, $C_1 = \left\{1+\lambda C_z^2 + \frac{\eta(\eta+2)\lambda C_x^2}{4} - 2\eta\lambda\rho_{zx}C_zC_x,\ C_2 = \frac{1}{R^2}\lambda C_x^2,\ C_3 = \frac{1}{R^{*2}}\lambda C_r^2\right\}$

$C_4 = \frac{1}{R}\lambda\{\eta C_x^2 - \rho_{zx}C_zC_x\}$, $C_5 = \frac{1}{R^*}\lambda\{\eta\rho_{xr_x}C_xC_r - \rho_{zr_x}C_zC_r\}$, $C_6 = \frac{1}{R}\frac{1}{R^*}\frac{\lambda\rho_{xr_x}C_xC_r}{2}$
$C_7 = \left\{1 + \frac{\eta(\eta+2)\lambda C_x^2}{8} - \frac{\eta\lambda\rho_{zx}C_zC_x}{2}\right\}$, $C_8 = \frac{1}{R}\frac{\lambda\eta C_x^2}{2}$, $C_9 = \frac{1}{R^*}\frac{\lambda\eta\rho_{xr_x}C_xC_r}{2}$
Differentiate equation (23) with respect to w_i $i = 1,2,3$ and equating it to zero, we have

$$w_1C_1 + w_2C_4 + w_3C_5 = C_7$$
$$w_1C_8 + w_2C_2 + w_3C_6 = C_8$$
$$w_1C_5 + w_2C_6 + w_3C_3 = C_9$$

In matrix notation these simultaneous equations can be written as

$$\begin{bmatrix} C_1 & C_4 & C_5 \\ C_4 & C_2 & C_6 \\ C_5 & C_6 & C_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} C_7 \\ C_8 \\ C_9 \end{bmatrix} \text{ or } Aw = c$$

Then

$$w = A^{-1}\boldsymbol{c}$$

The optimum values of $w_i, i = 1,2,3$ are given by

$$w_{1(opt)} = \frac{\Delta_1}{\Delta},\quad w_{2(opt)} = \frac{\Delta_2}{\Delta},\qquad w_{3(opt)} = \frac{\Delta_3}{\Delta}$$

Where,

$$\Delta = [C_1(C_2C_3 - C_6^2) - C_4(C_4C_3 - C_5C_6) + C_5(C_4C_6 - C_2C_5)]$$
$$\Delta_1 = [C_7(C_2C_3 - C_6^2) - C_4(C_3C_8 - C_6C_9) + C_5(C_8C_6 - C_2C_9)]$$
$$\Delta_2 = [C_1(C_3C_8 - C_6C_9) - C_7(C_3C_4 - C_5C_6) + C_5(C_4C_9 - C_5C_8)]$$
$$\Delta_3 = [C_1(C_2C_9 - C_6C_8) - C_4(C_3C_9 - C_5C_8) + C_7(C_4C_6 - C_2C_5)]$$

The resulting minimum MSE of $t_{(\eta)}$ is given by

$$MSE_{min}t_{(v)} = \bar{Y}^2[1 - M_8] \tag{24}$$

Where, $M_8 = \frac{(C_7\Delta_1 + C_8\Delta_2 + C_9\Delta_3)}{\Delta}$

Which is true if $0 < M_9 < 1$ and $\Delta > 0$

Percent relative efficiency (PRE) of the estimator $t_{(v)}$

Case (1); For $w_3 = 0$, The proposed estimator turns out to be the estimator proposed by Koyuncu et al. (2014) and Tiwari et al.(2022), which is given by

$$t_{v_1} = [w_1\bar{z} + w_2(\bar{X} - \bar{x})]\exp\left(\frac{\eta(\bar{X} - \bar{x})}{\bar{X} + \bar{x}}\right) \tag{25}$$

Where, w_1, w_2 are constants, whose values are obtained by minimizing MSE of t_{v_1} and η is suitable constant which takes real values.

4.2. Bias and Mean Square Error of the estimators t_{v_1}

To derive the Bias and MSE of the proposed t_{v_1} case first estimator under the first order of approximation, given by

$$Bias(t_{v_1}) = \bar{Y}\left[w_1\left\{1 + \frac{\lambda\eta C_x^2}{4}\left(1 + \frac{\eta}{2} - 2\rho_{zx}\frac{C_z}{C_x}\right)\right\} + w_2\frac{1}{R}\left\{\frac{\lambda\eta C_x^2}{2}\right\} - 1\right] \tag{26}$$

$$MSE(t_{v_1}) = \bar{Y}^2\left[1 + w_1^2\left\{1 + \lambda C_z^2 + \frac{\eta(\eta+1)\lambda C_x^2}{2} - 2\eta\lambda\rho_{zx}C_zC_x\right\} + w_2^2\frac{1}{R^2}\lambda C_x^2 - 2w_1\left\{1 + \frac{\eta(\eta+2)\lambda C_x^2}{8} - \frac{\eta\lambda\rho_{zx}C_zC_x}{2}\right\} - 2w_2\frac{1}{R}\frac{\eta\lambda C_x^2}{2} + 2w_1w_2\frac{1}{R}\lambda\{\eta C_x^2 - \rho_{zx}C_zC_x\}\right]$$

The expression MSE of t_{v_1} is given as

$$MSE(t_{v_1}) = \bar{Y}^2[1 + w_1^2M + w_2^2N - 2w_1O - 2w_2P + 2w_1w_2Q] \tag{27}$$

Where, $M = \left\{1 + \lambda C_z^2 + \frac{\eta(\eta+1)\lambda C_x^2}{2} - 2\eta\lambda\rho_{zx}C_zC_x\right\}$, $N = \frac{1}{R^2}\lambda C_x^2$, $Q = \frac{1}{R}\lambda\{\eta C_x^2 - \rho_{zx}C_zC_x\}$ $O = \left\{1 + \frac{\eta(\eta+2)\lambda C_x^2}{8} - \frac{\eta\lambda\rho_{zx}C_zC_x}{2}\right\}$, $P = \frac{1}{R}\frac{\eta\lambda C_x^2}{2}$

To obtain the values of $w_{1(opt)}$ and $w_{2(opt)}$

$$w_{1(opt)} = \frac{ON - QP}{MN - Q^2} \text{ and } w_{2(opt)} = \frac{MP - OQ}{MN - Q^2}$$

Putting these (w_{1opt}, w_{2opt}) in (27) The minimum MSE of t_{v_1} is given as

$$MSE_{min}(t_{v_1}) = \bar{Y}^2\left\{1 - \frac{MP^2 + NO^2 - 2\,OPQ}{MN - Q^2}\right\} \tag{28}$$

Case (2); For $w_2 = 0$, The proposed estimator will utilize the ranks only of the auxiliary variables X and the proposed class of estimator will be

$$t_{v_2} = [w_1\bar{z} + w_3(\bar{R}_x - \bar{r}_x)]\exp\left(\frac{\eta(\bar{X} - \bar{x})}{\bar{X} + \bar{x}}\right) \tag{29}$$

Where, w_1 and w_3 are suitably chosen constants to be determined such that MSE of t_{v_2} is minimum and η is suitable constant which takes real values.

4.3. The Bias and MSE of t_{v_2}

To derive the Bias and MSE of the proposed t_{v_2} case second estimator, up to the first order of approximation, is given by

$$Bias(t_{v_2}) = \bar{Y}\left[w_1\left\{1 + \frac{\lambda\eta C_x^2}{4}\left(1 + \frac{\eta}{2} - 2\rho_{zx}\frac{C_z}{C_x}\right)\right\} + w_3\frac{1}{R^*}\left\{\frac{\lambda\eta\rho_{xr_x}C_rC_x}{2}\right\} - 1\right] \tag{30}$$

$$MSE(t_{v_2}) = \bar{Y}^2\left[1 + Aw_1^2\left\{1 + \lambda C_z^2 + \frac{\eta(\eta+1)\lambda C_x^2}{2} - 2\eta\lambda\rho_{zx}C_zC_x\right\} + w_3^2\frac{1}{R^{*2}}\lambda C_r^2 - 2w_1\left\{1 + \frac{\eta(\eta+2)\lambda C_x^2}{8} - \frac{\eta\lambda\rho_{zx}C_zC_x}{2}\right\}\right.$$
$$\left. - 2w_3\frac{1}{R^*}\frac{\eta\lambda\rho_{xr_x}C_xC_r}{2} + 2w_1w_3\frac{1}{R^*}\lambda\{\eta\rho_{xr_x}C_xC_r - \rho_{zr_x}C_zC_r\}\right]$$

The expression MSE of t_{v_2} is given as

$$MSE(t_{v_2}) = \bar{Y}^2[1 + w_1^2A + w_3^2B - 2w_1C - 2w_3D + 2w_1w_3E] \tag{31}$$

Where, $A = \left\{1 + \lambda C_z^2 + \frac{\eta(\eta+1)\lambda C_x^2}{2} - 2\eta\lambda\rho_{zx}C_zC_x\right\} = M, B = \frac{1}{R^{*2}}\lambda C_r^2$
$C = \left\{1 + \frac{\eta(\eta+2)\lambda C_x^2}{8} - \frac{\eta\lambda\rho_{zx}C_zC_x}{2}\right\} = O, D = \frac{1}{R^*}\frac{\eta\lambda\rho_{xr_x}C_xC_r}{2},\ E = \frac{1}{R^*}\lambda\{\eta\rho_{xr_x}C_xC_r - \rho_{zr_x}C_zC_r\}$
To obtain the values of $w_1(opt)$ and $w_2(opt)$

$$w_{1(opt)} = \frac{CB - ED}{AB - E^2}, and\ w_{3(opt)} = \frac{AD - CE}{AB - E^2}$$

The minimum expression MSE of t_{v_1} is given as

$$MSE_{min}(t_{v_2}) = \bar{Y}^2\left\{1 - \frac{AD^2 + BC^2 - 2\,CDE}{AB - E^2}\right\} \tag{32}$$

5. Efficiency comparison

We present situation of efficiency comparison of the proposed class of estimator t_v with existing estimators, which is given as

(i) $MSE_{min}(t_v) < MSE(t_Y)$ if

$$\lambda C_z^2 + M_8 > 1$$

(ii) $MSE_{min}(t_v) < MSE(t_R)$ if

$$\lambda(C_z^2 + C_x^2 - 2\rho_{zx}C_zC_x) + M_8 > 1$$

(iii) $MSE_{min}(t_v) < MSE(t_{Reg})$ if

$$\lambda C_z^2(1 - \rho_{zx}^2)^2 + M_8 > 1$$

(iv) $MSE_{min}(t_v) < MSE(t_{GRR})$ if

$$\left[\frac{C_z^2(1 - \rho_{zx}^2)\lambda\{1 - \lambda C_x^2\}}{C_z^2(1 - \rho_{zx}^2)\lambda + \{1 - \lambda C_x^2\}}\right] + M_8 > 1$$

(v) $MSE_{min}(t_v) < MSE(t_{GER})$ if

$$\left[\left(1-\frac{1}{4}\lambda C_x^2\right)-\frac{\left(\left(1-\frac{1}{8}\lambda C_x^2\right)\right)^2}{1+C_z^2(1-\rho_{zx}^2)}\right]+M_8>1$$

(vi) $MSE_{min}(t_v) < MSE(t_{ER})$ if

$$\frac{1}{4}\lambda[(4C_z^2-4\rho_{zx}C_zC_x+C_x^2)]+M_8>1$$

(vii) $MSE_{min}(t_v) < MSE(t_{GR})$ if

$$\left\{\left(1-\frac{1}{4}\lambda C_x^2\right)-\frac{\left(1+\lambda\left\{\left(\mathrm{B}-\frac{1}{2\mathrm{A}}-\frac{1}{4}\right)C_x^2+\left(\frac{1}{2}-A\right)\rho_{zx}C_xC_z\right\}\right)^2}{1+\lambda\left[\left(2B-A-\frac{1}{4}\right)C_x^2+(1-2A)\rho_{zx}C_zC_x+(1-\rho_{zx}^2)C_z^2\right]}\right\}+M_8>1$$

Where, $M_8=\frac{(C_7\Delta_1+C_8\Delta_2+C_9\Delta_3)}{\Delta}$

6. Simulation Study

An empirical study is performed to show the performance of the proposed estimators in comparison of existing estimators. Four bivariate normal populations of size 1000 with different covariance matrices and mean vector $\mu = [2,2]$ are generated for this comparison, which gives the values of the study variable and the auxiliary variable X, R_x is the ranks of the auxiliary variable, the scrambling variable S is generated from normal distribution with mean 0 and standard deviation as10 percent of the standard division of X. The reported response is given by, $Z = Y + S$. The input data used to derive the four different populations and their descriptive statistics are given below.

Table 1 Data summary

	Population 1	**Population 2**	**Population 3**
N	1000	1000	1000
μ	$[2 \quad 2]$	$[2 \quad 2]$	$[2 \quad 2]$
Σ	$\begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}$	$\begin{bmatrix} 09 & 3.2 \\ 3.2 & 04 \end{bmatrix}$	$\begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}$
ρ_{zx}	0.3209	0.5154	0.6746
ρ_{zr_x}	0.3099	0.5039	0.6336
ρ_{xr_x}	0.9731	0.9738	0.9740

To calculate the percent relative efficiency (PRE), for the purpose of comparison of proposed class of estimator with other existing estimators. We conducted an empirical study and computed the percent relative efficiency (PRE) of the estimators $(t_Y,\ t_R,\ t_{Reg}, t_{GRR}, t_{GER}, t_{ER}, t_{GR}, t_{v_1}, t_{v_2},\ t_v)$ in comparison with t_Y . η is fixes at $-2, -1, 0, 1,\ 2$. For $n = 50,\ 100,\ 200\ and$ 300 the MSE of all the estimators are calculated using the derived formula. The following expression is used to obtain percent relative efficiency of different estimators with respect to mean estimator of the population mean,

$$PRE(t_\tau)=\frac{MSE(t_Y)}{MSE(t_\tau)}\times 100$$

Where, $\tau = Y, R, Reg, GRR, GER, ER, GR, v_1, v_2, v$

Table 2: PRE of different estimators with respect to t_Y for Population – 1

$N = 1000$	Estimators	$n = 50$	$n = 100$	$n = 200$	$n = 300$
	t_Y	100.00	100.00	100.00	100.00
	t_R	97.25	97.25	97.25	97.25
	t_{Reg}	110.18	110.18	110.18	110.18
	t_{GRR}	110.80	112.81	111.31	110.86
	t_{GER}	7.64	3.60	1.61	0.94
	t_{ER}	110.12	110.12	110.12	110.12
	t_{GR}	21.68	10.30	4.60	2.69
$\eta = -2$	t_{v_1}	111.80	110.92	110.51	110.37
	t_{v_2}	24.7	14.02	6.94	4.20
	$\boldsymbol{t_v}$	**176.70**	**176.71**	**176.65**	**176.64**
$\eta = -1$	t_{v_1}	113.90	111.96	110.28	110.64
	t_{v_2}	19.10	9.85	4.58	2.72
	$\boldsymbol{t_v}$	**134.50**	**132.99**	**132.28**	**132.01**
$\eta = 0$	t_{v_1}	115.69	112.76	111.34	110.86
	t_{v_2}	5.50	2.60	1.16	0.675
	$\boldsymbol{t_v}$	**120.02**	**117.12**	115.68	**115.19**
$\eta = 1$	t_{v_1}	116.38	113.10	111.47	110.92
	t_{v_2}	7.43	4.89	2.71	1.72
	$\boldsymbol{t_v}$	**116.40**	**113.12**	**111.51**	**110.96**
$\eta = 2$	t_{v_1}	115.80	112.80	111.34	110.86
	t_{v_2}	31.60	31.31	30.75	30.05
	$\boldsymbol{t_v}$	**120.80**	**118.21**	**116.98**	**116.57**

Table 3: PRE of different estimators with respect to t_Y for Population – 2

$N = 1000$	Estimators	$n = 50$	$n = 100$	$n = 200$	$n = 300$
	t_Y	100.00	100.00	100.00	100.00
	t_R	126.82	126.82	126.82	126.82
	t_{Reg}	131.28	131.30	131.28	121.28
	t_{GRR}	136.91	133.90	132.45	131.97
	t_{GER}	8.04	3.80	1.68	0.99
	t_{ER}	126.90	126.90	126.90	126.90
	t_{GR}	1.33	0.75	0.36	0.22
$\eta = -2$	t_{v_1}	132.40	131.80	131.52	131.42
	t_{v_2}	20.67	10.96	5.20	3.09
	$\boldsymbol{t_v}$	**391.26**	**383.69**	**380.27**	**379.18**
$\eta = -1$	t_{v_1}	134.75	132.92	132.00	131.71
	t_{v_2}	14.92	7.34	3.35	1.97
	$\boldsymbol{t_v}$	**198.83**	**198.03**	**197.63**	**197.50**
$\eta = 0$	t_{v_1}	136.78	133.89	132.45	131.71
	t_{v_2}	5.50	2.61	1.16	0.68
	$\boldsymbol{t_v}$	**155.02**	**152.12**	**150.68**	**150.20**
$\eta = 1$	t_{v_1}	137.62	134.28	132.61	132.05
	t_{v_2}	0.32	0.68	0.08	0.04
	$\boldsymbol{t_v}$	**139.83**	**136.32**	**134.59**	**134.01**
$\eta = 2$	t_{v_1}	136.92	133.92	132.45	131.97
	t_{v_2}	15.08	11.50	7.34	5.02
	$\boldsymbol{t_v}$	**138.07**	**135.30**	**133.94**	**133.50**

Table 4: PRE of different estimators with respect to t_Y for Population – 3

$N = 1000$	Estimators	$n = 50$	$n = 100$	$n = 200$	$n = 300$
	t_Y	100.00	100.00	100.00	100.00
	t_R	158.94	158.94	158.94	158.94
	t_{Reg}	170.30	170.30	170.30	170.30
	t_{GRR}	176.46	173.20	171.59	171.05
	t_{GER}	9.35	4.42	1.96	1.14
	t_{ER}	130.42	130.42	130.42	130.42
	t_{GR}	2.41	1.17	0.53	0.32
$\eta = -2$	t_{v_1}	173.11	171.64	170.88	170.64
	t_{v_2}	27.70	16.80	8.68	5.35
	$\boldsymbol{t_v}$	**349.01**	**348.80**	**348.68**	**348.65**
$\eta = -1$	t_{v_1}	175.02	175.54	171.29	170.88
	t_{v_2}	17.60	8.86	4.07	2.40
	$\boldsymbol{t_v}$	**267.12**	**265.85**	**265.23**	**265.03**
$\eta = 0$	t_{v_1}	176.40	173.19	171.58	171.05
	t_{v_2}	6.09	2.88	1.28	0.748
	$\boldsymbol{t_v}$	**227.20**	**223.98**	**222.38**	**221.84**
$\eta = 1$	t_{v_1}	176.92	173.43	171.69	171.11
	t_{v_2}	0.61	0.30	0.13	0.08
	$\boldsymbol{t_v}$	**203.70**	**199.58**	**197.53**	**196.85**
$\eta = 2$	t_{v_1}	176.46	173.20	171.59	171.05
	t_{v_2}	0.28	0.15	0.07	0.05
	$\boldsymbol{t_v}$	**188.59**	**184.58**	**188.61**	**181.95**

7. Interpretations of the computational results

The results are presented in Tables 2, 3 and 4 for the considered population 1, 2, 3 respectively. It is clear from these tables that the percent relative efficiency of proposed estimator t_v is better than t_{v_1}, t_{v_2} for population1, 2 and 3 respectively. That means using ranks of the auxiliary variable X additionally with information on auxiliary variable X improvised the estimator in terms of percent relative efficiency. The proposed estimator is also outperforms with respect to the other existing estimators $(t_Y,\ t_R,\ t_{Reg}, t_{GRR}, t_{GER}, t_{ER}, t_{GR})$ also. For, the values η, it can be seen from Tables 2, 3 and 4 that the proposed estimator has the maximum percent relative efficiency for $\eta = -2$, which is 176.70, 391.26 and 349.01 respectively, and it decreases as we move from $\eta = -2$ to $\eta = 2$. Ranks alone are not giving is better percent relative efficiency, can be seen for all values for t_{v_2}.

8. Conclusion

In this study, the estimation of population mean for a sensitive study variable using additive RRT model is suggested using single or dual auxiliary variable and simple random sampling. For improving the estimator we assigned rank to our auxiliary variable related to the study variable. When comparing with the existing estimator, they provide better results in terms of percent relative efficiency. This confirms that the easily available additional information on ranks without any cost helps in improving the efficiency of estimator.

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